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ESSAY

ON

INVOLUTION & EVOLUTION;

Particularly applied to the

Operation of Extracting

THE

ROOTS OF EQUATIONS AND NUMBERS,

ACCORDING TO

A PROCESS ENTIRELY ARITHMETICAL;

Superseding, by its greater Simplicity, Swiftness, and Regularity, every other Method that has yet been attempted.

A New Edition;

WITH

A POSTSCRIPT,

Vindicating the Claims of the Author in the maturing and bringing the Subject to Perfection; and showing the vast Superiority of his Demonstrations and Methods to those which Mr. HOLDRED has published since the first Edition of this Essay.

TOGETHER WITH

AN APPENDIX

ON

Figurate Numbers and Arithmetical Equivalents:

The whole being adapted to the

SKILFUL ANALYST AND THE EXPERT ARITHMETICIAN.

BY P. NICHOLSON,

Author of "An Introduction to Increments," "Combinatorial Essays," "Rudiments of Algebra," "Architectural Dictionary," &c. &c.

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☞ The following Letter, received by the Author, will show the degree of Approbation with which the First Edition of this Work has been received among foreign Mathematicians :—

"Institut de France,

"ACADEMIE ROYALE DES SCIENCES.

" *Paris, le* 10 *Juillet*, 1820.

" *Le Secrétaire perpétuel de l'Académie à Monsieur*
" *Nicholson.*

" *L'ACADEMIE a reçu, Monsieur, avec un*
" *vif intéret l'Essai que vous avez bien voulu lui*
" *adresser sur l'Involution et l'Evolution, ou Méthode*
" *pour déterminer la Valeur Numérique d'une Fonc-*
" *tion quelconque d'une Quantité inconnue. Elle*
" *me charge de vous remercier en son nom de l'envoi*
" *de cet ouvrage intéréssant qu'elle a fait déposer*
" *honorablement dans la Bibliothèque de l'Institut,*
" *et de vous en témoigner sa reconnoissance.*

" *Reçevez je vous prie, Monsieur, l'assurance de*
" *ma consideration la plus distinguée.*

" B. G. CUVIER."

[Translation.]

[Translation.]

French Institute,
ROYAL ACADEMY OF SCIENCES.

Paris, July 10, 1820.

The Perpetual Secretary of the Academy to Mr. Nicholson.

SIR, the Academy has received, with a lively degree of interest, the Essay that you obligingly addressed to it on Involution and Evolution, or a Method of determining the Numerical Value of any Function of an unknown Quantity. I am desired, in its name, to thank you for sending this interesting work, which has been honourably placed in the Library of the Academy; and to express the sense of obligation which the Institution entertains for your attention.

Receive, Sir, I beg, the assurance of the most distinguished respect with which I am,

Your's, &c.

B. G. CUVIER.

ERRATA, Page 71.

CASE 2.—When the two numbers have different signs, *Multiply the figure in the unit's place of the multiplicand by the digit; subtract the product from the figure in the unit's place of the other number, if less than the figure from which it is to be taken; but if this product be greater, increase the figure in the unit's place by the least number of tens that will make it equal to or just greater than the said product; subtract the product from the digit thus increased; write the remainder in the unit's place of the number required. Proceed in the same manner to find the figure in the place of tens, observing to carry as many units as the number of tens borrowed; and so on, from figure to figure, till the whole is completed.*

INTRODUCTION.

Mr. Theophilus Holdred, a gentleman but little known in the mathematical world, some time since submitted for my inspection and opinion an original tract, containing a method of finding the roots of equations of all degrees in numbers; but from the obscurity, want of connexion, and the antiquated manner in which the subject was treated, I was able to form but a very imperfect idea of the principles upon which his method was founded.

Anxious, however, to accomplish what had been deemed by the first mathematicians a matter of the utmost importance, I resolved not to lose sight of so desirable an object. Without attending to the manner in which Mr. Holdred had considered the subject, but keeping steadily in view Newton's principle of approximation, I soon conceived that, to extract the root of an equation in numbers, it was requisite to find a series of transformed equations, of such a nature, that in every two consecutive equations the root of the former should be diminished by a single digit or denomination, which should have the greatest local value possible, not exceeding the root; and also that the process of transformation should be performed by arithmetical rules, instead of the binomial theorem, as had hitherto been practised.

Having discovered the manner in which these desirable ends were to be obtained, together with a demonstration of the theory of the method, I communicated the result to

Mr. Holdred, who admitted the simplicity of the principle; allowing it to be more concise and easier of comprehension than his own, and that it led immediately to the rule, without circuitous steps in the demonstration: at the same time he acknowledged that he entertained no idea of his rule being derived from any established principles of transformation. However, upon a further examination of his method, I perceived that it was founded upon the same principle as that upon which Raphson has founded his method of approximation; viz. that of always referring to the original equation. Considering what I had done as an improvement, he agreed to add it, by way of an Appendix, to the tract which he proposed publishing.*

* As a proof of what is asserted above, the following note is a part of his original manuscript. After having shown how to extract the cube root, he begins thus: "In like manner the root of a mixed equation may be extracted. Let the equation be $a^3+ba^2+ca=N$," where a is used instead of x; "take the letters $g+y$ for a, and raise this value of a to all the powers of a in the given equation, and mulliply by the respective coefficients, and we get these following equations:

$$\begin{aligned} a^3 &= g^3 + 3g^2y + 3gy^2 + y^3 \\ ba^2 &= bg^2 + 2bgy + by^2 \\ ca &= cg + cy \\ \hline \text{The sum is } N &= A + By + Cy^2 + y^3 \end{aligned}$$

By putting B for the sum of the coefficieots of y, C for the sum of the coefficients of y^2, and A for the sum of the powers and multiples of g, under which it stands, A is therefore the subtrahend; then, by transposition, $N-A=By+Cy^2+y^3$, which is the resolvend. Therefore $B+C$ is the first imperfect divisor. Then put e for the second figure of the root; and, making $r=g+e$, raise this equation to all the powers of a in the giveo equation, and multiply by the respective coefficients, and we get the followiog equations:

$$\begin{aligned} r^3 &= g^3 + 3g^2e + 3ge^2 + e^3 \\ br^2 &= bg^2 + 2bge + be^2 \\ cr &= cg + ce \\ \hline A' &= A + Be + Ce^2 + e^3 \end{aligned}$$

Here

After I had pointed out many defects and obscurities in his manuscript, he agreed to write the whole anew, under my inspection, adopting such further improvements as might occur during the period of re-writing it. The improvements produced, at last, an entirely new form in his practical operations.

Here A is put for the sum of the powers and multiples of r, under which it stands; then, by transposition, we get

$$A'-A=Be+Ce^2+e^3.$$

This is the next subtrahend, which, being taken from the first resolvend, leaves the new resolvend $N-A'$. Then, putting u for the remaining part of the root, we have $a=r+u$. This equation being raised to all the powers of a in the given equation, and multiplied by the respective coefficients, we get the following equations:

$$\begin{array}{l} a^3=r^3+3r^2u+3ru^2+u^3 \\ ba^2=br^2+2bru+bu^2 \\ ca=cr+cu \\ \hline \text{The sum } N=A'+B'u+C'u^2+u^3. \end{array}$$

Here B' and C' are put for the sums of the coefficients of the respective powers of u, under which they stand. Then, by transposition, we get $N-A=B'u+C'u^2+u^3$; this is the resolvend, the same as was found by taking the subtrahend from the first resolvend.

Therefore $B'+C'$ is the new imperfect divisor. Now, to get the value of B' in terms of the first divisor; because $B'=3r^2+2br+c$, and $r=g+e$, raise this value of r to all the powers of r in the equation, which gives the value of B', and multiply by the respective coefficients, and the following equations are had:

$$\begin{array}{l} 3r^2=3g^2+6ge+3e^2 \\ 2br=2bg+2be \\ c=c \\ \hline B'=B+2Ce+3e^2 \end{array}$$

And because $C'=3r+b$, therefore, by proceeding in the like manner, we get the following equations:

$$\begin{array}{l} 3r=3g+3e \\ b=b \\ \hline C'=C+3e. \end{array}$$

Thus,

I took every opportunity of recommending his work and procuring him subscribers for the publication of it. To this end I announced it in No. ccxlvii of Tilloch's Philosophical Magazine, for November 1818; and I also alluded to it in my Combinatorial Essays, published in the same year.

When I had finished the paper containing my demonstration, with the rules and examples that were to be subjoined to his work, I found that Mr. Holdred, certainly in opposition to his real interests, had been persuaded by an acquaintance, an utter stranger to algebraic operations, to publish his own manuscript; rejecting my improvements, on the supposition that what I had done might diminish the credit of his own performance, unless I would allow him to let them pass as his own.

At this treatment I could not but feel greatly indignant; and hearing that a new book on algebra, containing rules and methods for extracting the roots of equations of all dimensions accurately, was in the press; and having been at considerable pains in composing the Appendix for Mr. Holdred's tract; without regarding what his intentions might be, I resolved to insert what I had done in a new work entitled Rudiments of Algebra, which I was then preparing for publication.

Thus, if B and C are the sums of the respective coefficients of the unknown parts of the root, $B+Ce+e^2$ will be the first true divisor, which being multiplied by e (the quotient figure), gives the subtrahend; and the new imperfect divisor is formed out of the former, in the like manner as in extracting the single cube root; doing with B, in this case, as by the treple square of the root in that case; and by C, in this case, as by the treple root in that case. Having got the value of B' and C', which makes the new imperfect divisor, proceed, in all respects, as before."

My Rudiments were printed early in July 1819, and it is remarkable that on the 1st of the same month was read to the Royal Society a paper written by Mr. Horner, of Bath, and containing the demonstration of a method of finding the roots of equations of all degrees by continuous approximation. This circumstance, however, remained unknown to me till the 8th of the December following the publication of my work. The Volume or Part of the Philosophical Transactions which contained Mr. Horner's paper was published Dec. 1st, 1819. I read the article attentively, and, though my mind had been so long devoted to the subject, I did not at that time fully comprehend the force of his reasoning. I perceived, however, that the paper contained the substance of what I had previously written and published, and that, in addition thereto, he had shewn the means of performing the operation without using the figurate factors.

But as the demonstration can be understood only by the very few who have been initiated into the algorithm of the calculus which he has employed, and as he has given no rules in words for performing his operations, nor any examples calculated to elucidate his meaning, I resolved to try, without collateral assistance, to derive all that he had done from principles well known to every algebraist. I have succeeded in my wishes, without receiving any aid from this gentleman's labours beyond the knowledge of the possibility of the improvement introduced by him.

Mr. Horner has nowhere observed that the method he has given is *complete;* and his calling it a method of continuous approximation serves to confirm the idea that he was not aware of it being any thing more than an approximating rule. If his rule be perfect, the appellation which he has given it conveys a meaning short of its merits; and

that it can easily be rendered complete will be evident to any one who makes himself master of the very simple demonstration I have given in the following pages.

He has said that there is no advantage in applying his method of continuous approximation to the integral part of a root; he therefore directs us to transform every such equation into another, of which the root shall be less than the root of the original equation by the integral part of that root. In this, however, he is mistaken, as will appear from the example given in page 44, where all the denominations of the integral part of the root, after the third, are found as easily, and with as much certainty, as the decimal part of the same.

His general formula and examples are defective, from his not explaining how the root of an equation may be extracted when the highest power is accompanied with a numeral coefficient other than unity.

Another omission consists in his not having given any rule in words by which the operation may be performed. And as none of the examples exhibit more than one complete step of the work, and that not in the most obvious manner, it appears to me that the bare contemplation of his formula, and the examples given as illustrations of the principle, would not enable a person, unacquainted with the symbolical language of algebra, to form a rule for himself sufficient to direct any future operation.

It is by no means my wish to detract from the merits of this gentleman: his endeavours have been successful in the discovery of an original and excellent method of extracting the roots of equations, altogether differing from that invented by Mr. Holdred, though capable of being derived from it, as will be shewn hereafter.

Had there been no such blemishes as I have pointed out,

there would have been no good grounds for publishing my labours at this time; and my efforts to explain the method upon more evident principles, and perform the operations by more regular processes, would have been altogether unnecessary.

With respect to the contracted operation for extending the roots of equations by approximation, though first made public by Mr. Horner, it had been done by Mr. Holdred many years prior to its insertion in the Philosophical Transactions.

It is impossible for me to decide which of these gentlemen is entitled to the honour of priority of invention; I shall content myself with stating what I know of the matter, and leave this point to be ascertained hereafter. Both seem to me fairly entitled to be considered as original inventors, as the principles and modes of demonstration are so different from each other.

I have not the pleasure of being acquainted with Mr. Horner, nor do I know how long he may have been in possession of his method*.

Mr. Holdred I have known about ten years. At the commencement of our acquaintance he shewed me his general method of extracting the roots of equations in numbers; and, if I remember rightly, he said that he had been in possession of it for many years. I am informed that his long-projected work is now in the press; and from some conversation I had with him before our difference

* I am informed by Mr. Dickson, that about twelve months ago he purchased, at his shop in St. Martin's-le-Grand, an Essay on the Numerical Solution of Equations, by Budan; at which time he mentioned that he was engaged expressly upon this subject.

took place, it is most likely that he will inform the public of the circumstances that led to the discovery of the principles of his method.

I have had no communication with him since his determination to leave out the Appendix which I had prepared for his work. My sole desire in taking up the subject was to benefit him, and promote the general interests of the science. But we view the object of our difference from opposite points; and, in all probability, our acquaintance is for ever terminated. I feel, however, that what I have done since was but justice to myself, in consideration of the indignity offered to me by his rejecting the improvements which I communicated to him, and which he had pledged himself to adopt. I consider, too, that by publishing the method, I have secured his credit as an original inventor, which otherwise might have been disputed, and the sole honour ascribed to Mr. Horner.

It may not be improper to insert here a note taken from the Paper in the Philosophical Transactions before alluded to. It will shew the Reader the value which Mr. Horner attaches to the invention, and consequently the honour which he conceives himself entitled to receive. It is as follows:—

"The only object proposed by the Author in offering this Essay to the acceptance of the Royal Society, for admission into the Philosophical Transactions, is to secure, beyond the hazard of controversy, an Englishman's property in a useful discovery. Useful he may certainly be allowed to call it, though the produce of a purely mathematical speculation; for of all the investigations of pure mathematics, the subject of *approximation* is that which comes most directly, and most frequently, into contact with the practical wants of the calçulator.

"How far the manner in which he has been so fortunate as to contemplate it has conduced, by the result, to satisfy those wants, it is not for him to determine; but his belief is, that both Arithmetic and Algebra have received some degree of improvement, and a more intimate union. The abruptness of transition has been broken down to a gentle and uniform acclivity."

In treating the subject, I have all along endeavoured to place the evidence of the rules in so clear a light as to be intelligible to any attentive reader, even of ordinary capacity.

The improvements which I have made upon Mr. Holdred's, or the figurate method, principally consist in giving a new demonstration of the theory; in introducing an entirely different form of operation, in which the relation of the steps is clearly seen, and the coefficients of the new transformed equations exhibited; and in disembarrassing the process from algebraic symbols. The work is made to consist of uniform steps, comprising classes of numbers, derived from each other, and from the preceding steps in the simplest manner, and according to rules purely arithmetical: thus the law of the operation is rendered obvious to the eye and intelligible to the understanding.

A further improvement which I have introduced into the demonstration, arises from my having changed the order of the coefficients. This alteration prevents their being reverted in the practical operations, as is the case in Mr. Horner's arrangement, and in that which I had introduced into my Rudiments of Algebra before his method was made public.

In the first of Mr. Holdred's manuscripts the incipient step of every operation was to transform the proposed equa-

tion into another, before his method could be applied. The remaining steps were contained in one column only; and each step was derived from the preceding steps, and from algebraic equations expressing the values of the new coefficients: a mode extremely inconvenient in arithmetical operations.

Here, too, I must observe, that Mr. Holdred never used any coefficient to the highest power other than unity; and this also is the case with Mr. Horner. Nor did Mr. Holdred ever give any rule in words, except for extracting the single cube root, as he terms it*.

In the following little Treatise I have given two distinct methods for extracting the roots of equations; the first is the figurate, with the improvements that I have enumerated; the second is the non-figurate, and is, in practice,

* It is certainly remarkable that Mr. Holdred, who is a watch-maker by trade, and has spent but little of his time in the study of mathematics, and who, not being conversant with the modern improvements in the science, is unacquainted with some of the elementary principles of algebra, should have been the first (as appears probable from his age, and from the time I have known him) to have made the discovery of a general method of extracting the roots of equations; a discovery which has always been considered as a *desideratum*, and which has been attempted, unsuccessfully, by the most eminent mathematicians of Europe.

He was not acquainted with any work on algebra later than Ward or Ronaynes; and so partial was he to the old forms of notation, viz. that of employing the vowels *a*, *e*, *i*, *o*, *u*, &c. instead of *x*, *y*, *z*, that I found it difficult to persuade him to introduce the final letters for unknown quantities in the second manuscript which he wrote under my inspection.

In short, his first manuscript remained in his hand without any alteration that could be called an improvement; and to a reader of modern works nothing can appear more uncouth than his manner of treating the subject. However, as there are still many traces of the first manuscript to be perceived in the second, and as this last is about to be published, the reader will have an opportunity of judging for himself in this matter.

the same as that published in the Philosophical Transactions by Mr. Horner. I have also shewn that this latter method is easily deduced from the figurate previously explained; and that the whole depends solely upon the binomial theorem and the known properties of figurate numbers. By avoiding the introduction of the differential and functional calculus, and by deriving both methods from the common principles of algebra, I conceive that important advantages are gained. By these means the subject is rendered more generally intelligible, and the transition from symbols to the arithmetical operation is easier and more natural.

I have likewise inserted two different demonstrations. In the first I have shewn how any mixed powers may be involved by a direct process, when the root is given; and, by the converse of the same principle, how the roots of equations of all degrees may be extracted without recurring to any other principles. The second demonstration is derived from the consideration of the process of transformation, as it occurred to me at first. These two demonstrations, though very differently conceived, are performed in nearly the same manner; and Mr. Horner's refinement on the figurate principle may be derived, in the same manner, from either of them, as I have exemplified with respect to the non-figurate method.

It is surprising that the general method of extracting the roots of equations in numbers should not have been discovered long ago, since it is so simple as to be a mere corollary to any transformed equation; but writers on algebra seem, in general, to suppose that such a method must consist in the invention of a formula requiring only the knowledge of extracting the roots of pure numbers or powers, as has been done by Cardan and Descartes.

Let any one compare the general method here given with the particular formulas hitherto discovered, or even with the best modes of approximation, and he will soon be convinced of the decided superiority of this new method.

The formula discovered by Scipio Ferreus, but first published by Cardan, for extracting the roots of cubic equations, requires the second term to be exterminated; and after this preparation, it applies only to those equations that have two impossible roots. The labour in the subsequent operations is immense; for the several numbers in the first branch of the operation must be reduced to one, before the cube root can be extracted; and in this reduction, the square root of the second number must be found to three times the places of figures that are required in the root. Again, the second branch, which is to be added to the first, is obtained by dividing the *n*th part of the coefficient of the second term (*n* being the exponent of the highest power) by the result of the first branch, to as many figures, at least, as are wished for in the root of the proposed equation.

The remainders never terminate; for even when the equation admits of an exact root, this is only indicated by a series of nines.

Similar observations may be made on the formulas of Descartes, and other writers.

The rules for finding the roots of equations by means of sines and tangents, though they extend to the cases where Cardan's formula does not apply, and are something less laborious in their application, still they are attended with much more trouble than the rules of the new method, and are foreign to the purposes of pure algebra, as they require a previous knowledge of trigonometry.

The processes for finding the roots of equations by ap-

proximation, either upon Newton's or Raphson's principle, are general; they require no previous preparation, and are rapid compared with any of the methods indicated by general formula. Still they are clogged with algebraic symbols, uncertain as to the number of accurate figures obtained, and much less rapid in ascertaining the root than the operations performed by the new method, where no factors are used except single digits, where every figure of the root is ascertained with certainty, and where every remainder will terminate when the equation has an exact root.

From what has been stated respecting the principle which I have employed in the demonstration of the new process, it appears that this method also applies to exterminating the second term of an equation of n dimensions, and to finding a number greater than the greatest, or less than the least, root of an equation.

As these problems will be useful in the discovery of the first figure or two of the root in equations which contain negative coefficients, or in which the absolute number is negative, I shall illustrate them here by way of preparation. And first, for the sake of those who have reflected but little on these subjects, I shall explain their precise signification.

To *transform* an equation is to find another, of which the root shall be greater or less than that of the proposed equation by a given number.

To *exterminate the second term* of an equation of n dimensions, is only to find a new equation, of which the root shall be greater or less than that of the one proposed, by the nth part of the coefficient of the second term.

To find a root greater than the greatest root of an equation, is to find a new equation, of which the root shall be greater or less than that of the one proposed, by such a

number as will make all the terms of the transformed equation affirmative.

Hence the very same rule will apply to transform an equation, to exterminate the second term, and to find a number greater than the greatest root of an equation, as applies to the extraction of the root of an equation in numbers.

To illustrate these propositions, let it be required to transform the equation $x^3-9x^2+7x+12=0$ into another that shall want the second term.

Divide the coefficient 9 of the second term by 3, and the quotient is 3; then perform the marginal operation according to the Formula in p. 36, and we shall have 1, 0, —20, and —21 for the coefficients and absolute number of the new equation; so that if we represent the new unknown quantity by v, the transformed equation will be $v^3-20v-21=0$.

1	—9	7	12
	3	—18	—33
	—6	—11	—21
	3	— 9	
	3	—20	
	0		

To find a number greater than the greatest root of the equation $x^3-5x^2+7x-1=0$.

Here it is evident that no number less than one-third of the coefficient will effect this purpose. Let us first try 2, which is something greater than one-third, and we shall have the result exhibited in the incomplete operation No. 1. As we already perceive that the sum —3 of the last column is negative, we must try a number higher than 2. Making the experiment with 3, we have the marginal operation; and as all the coefficients are now affirmative, as well as the absolute number, 3 is greater than the greatest root of the proposed equation.

No. 1.

1	—5	7	—1
	2	—6	—2
	—3	—1	—3

1	—5	7	—1
	3	—6	3
	—2	1	2
	3	3	
	1	4	
	3		
	4		

In transforming an equation to another, if the coefficients of the last term, or absolute number of the proposed and transformed equations, have different signs, it is evident that the root must have some intermediate value between zero and the number by which the original root has been augmented or diminished.

The following observations will serve to give an idea of the comparative merits of the two methods for extracting the roots of equations, which are given in the subsequent pages.

In extracting the root of an equation of n dimensions according to the figurate method, the increase of figures at each step may be expressed by the formula $\frac{(n-1)^{3|1}}{1^{3|1}}+3(n-1)$, which, in equations of the second, third, fourth, fifth, sixth, and seventh degrees, will produce at each step a constant increase of 4, 10, 19, 32, 50, and 74 figures respectively.

The increase of figures at each step of the operation, according to the non-figurate method, may be expressed by the formula $\frac{n^{3|1}}{1^{3|1}}-1$, which, in equations of the second, third, fourth, fifth, sixth, and seventh degrees, produces at each step a constant increase of 3, 9, 19, 34, 55, and 83 figures respectively; so that in this latter method the increase is less in quadratic and cubic equations, equal in biquadratic, and greater in higher equations, than in the first method.

Again, in extractions not exceeding the third degree, performed by the aid of figurate factors, no marginal operations are necessary; but in equations of higher dimensions subsidiary operations are indispensable. Besides, the figurate

factors themselves consist of more than one place of figures, and become larger as the equation increases in the magnitude of its exponent.

In extracting a root by the non-figurate method, there is, as we have shewn, a much greater increase of figures at each step after the fourth degree than in the figurate method; and the trouble of multiplying a number by a digit, and at the same time adding the product to another number, is much greater than that of merely multiplying the numbers of each class by the new figure of the root, as practised in the figurate method. But since in the non-figurate process no marginal operations are ever required, the advantage certainly remains with the non-figurate method; and this will appear more decidedly the case as the dimension of the equation increases.

By either of these methods the increase of figures at each step is a constant number; but in extracting the root of an equation of any degree by the old modes of approximation, the increase of figures at each step is variable, and is greater in proportion to the number of figures ascertained in the root.

It is by no means easy to exhibit a formula for ascertaining the increase of figures at each step in extracting any proposed root of a *number* by the old methods. However, I shall assign two formulas, which serve for the cube and biquadrate roots.

The increase of figures in which the mth step of the work exceeds the $(m-1)$th step in extracting the cube root will be $2(m+8)$. Hence the fourth step will exceed the third by 24 figures. Again, the fifth step will exceed the fourth by 26 figures; while by the most improved method the constant increase is only 9 figures.

In extracting the fourth root of a number by the common method, the number of figures by which the mth step exceeds the $(m-1)$th step is $6m+25$. Hence, making $m=4$, the fourth step will exceed the third by 49 figures; and by making $m=5$, the fifth step will exceed the fourth by 55 figures; while, according to the new method, the constant increase is only 19 figures.

How greatly superior, then, is the new general method of evolution, even when applied to the particular case of extracting the roots of numbers, when compared to the old rules for this express purpose! It must be observed, that the two last formula include the figures which are usually erased, as well as those which are exhibited.

The contracted process, which is used as an appendage to the regular operation in the new method, is a very great acquisition. If r be the number of figures ascertained by the regular rule, and s the number of figures in the first subtrahend, then, in an equation of the nth degree, the whole number of figures in the root, obtained by the regular and contracted rules conjointly, will be expressed by the simple formula $s+(r-1)n$. By way of illustration, let n vary, while $s=4$ and $r=3$; and the formula will become $4+2n$; then if n be taken successively equal to 2, 3, and 4, the whole number of figures in the root will be 8, 10, and 12 respectively. So that the advantage increases in proportion to the magnitude of the exponent of the root which we desire to extract.

In extracting the root of an equation to R correct places of figures, it is not necessary to find every figure by the regular rule; for after r places are found by the operation at length, the remaining figures may be ascertained by the contracted process. As, for the sake of expedition, it is of importance to know how many places must be found

at length, I shall for that purpose assign the formula $r=\frac{R-s}{n}+1$; where R represents the number of true places required in the root, s the number of figures in the first subtrahend, and n the dimension of the equation.

As an example, Let it be required to extract the root of a biquadratic equation to 14 correct places of figures. Here $R=14$, $n=4$, and suppose $s=2$; then the formula becomes $\frac{14-2}{4}+1=3+1=4$; whence 4 is the number of figures which must be found by the operation at length. This result agrees with Example V, page 32, and Example XII, page 48, from whence the data were taken.

It is rather singular that Mr. Horner has not applied the non-figurate method to finding the limits of the roots of equations, and the number of impossible roots. I shall show how easily this might have been done by applying it to the very same example as that in which he has introduced the figurate method*.

EXAMPLE.—Has the equation $x^4-4x^3+8x^2-16x+20=0$ any real roots?

To solve this question, *we must find a series of transformed equations, such that, at each successive operation, the root shall be diminished by unity; then on examining the coefficients of the transformed equations, for every value of* x *in which we find zero placed between two numbers having the same sign, we may conclude that the equation contains one*

$x=0$, 1	−4	8	−16	20
0	1	−3	5	−11
1	−3	5	−11	9
	1	−2	3	
	1	−1		
	1			
$x=1$, 1	0	2	−8	9
0	1	1	3	−5
1	1	3	−5	4
	1	2	5	
	1	3		
	1			
$x=2$, 1	4	8	0	4

* It must be observed that Mr. Horner does not find the coefficients of the new equation as I have done, but he finds the divisors at once.

pair of impossible roots. Thus the above operation exhibits two pair of impossible roots; consequently in the proposed biquadratic equation none of the roots are possible.

In applying the above process, it is useless to proceed farther than the greatest limit.

Hence it may be inferred, that in any given or tranformed equation, when the exponents of any two adjacent terms differ by 2, and when the signs of these terms are identical, the equation will have two impossible roots.

Before the student proceeds to the extraction of roots, it will be necessary for him to understand something of the nature of their limits; whether the roots are possible or impossible, or whether they are affirmative or negative, in order that he may begin the operation. I shall, therefore, give the following sketch:

Quantities which are limits to the roots of an equation, if substituted for the unknown quantity, give results alternately affirmative and negative.

If the results arising from the substitution of two quantities be both affirmative or both negative, either no root of the equation, or any even number of roots, lies between them; but if these results have contrary signs, an odd number of roots must lie between them.

Therefore, if a series of quantities can be found which give as many results, alternately affirmative and negative, as the equation has dimensions, these must be the limits to the roots of the equation; and because an odd number of roots lies between each two succeeding terms of the series, and as there are as many terms as the equation has dimensions, this odd number cannot exceed unity.

If the last term of an equation of any even number of dimensions be negative, the equation will at least have two possible roots, one affirmative and the other negative.

Since the roots of every quadratic equation must either be both possible or both impossible, and since every equation of a higher dimension may be supposed to be formed of quadratics, impossible roots will enter the higher equations in pairs; therefore every equation of an odd dimension will, at least, have one possible root, and the number of real roots must always be odd; but when the exponent of the root is even, the number of real roots must also be even, as well as the number of odd roots, or they may be all possible or impossible.

With regard to the number of affirmative or negative roots, *every equation whose roots are possible has as many changes of signs from + to —, and from — to +, as it has affirmative roots, and the remaining ones, if any, negative.*

Thus the simple equation $x-a=0$ has one change of signs, + —; and therefore by this rule, as it has but one root, that root is affirmative, which we already know to be the case.

The quadratic equation $x^2-ax+b=0$ has two changes of signs, + —, — +; and as it has two roots, both are affirmative, otherwise they are impossible; and this is actually the case when b is greater than $\frac{a^2}{4}$.

The quadratic equation $x^2+ax+b=0$ has no change of signs, and therefore no affirmative root.

When the last term of a quadratic equation is negative, both roots are always possible. Therefore the quadratic equation $x^2+bx-a=0$, having one change of signs + — in the second and third terms, has always one affirmative root, and the other negative; likewise, the quadratic equation $x^2-ax-b=0$, having one change of signs + — in the first and second terms, has always one affirmative root, and the remaining one negative.

If one of the terms of an equation be wanting, between two terms which have unlike signs, the rule will apply as if it were complete; thus, the cubic equation $x^3-6x-2=0$, being considered complete, has only one change of signs $+\ -$ in the first and second terms; for if when complete, whether the second term be considered affirmative or negative, it will still have only one change; thus, $x^3+0x^2-6x-2=0$ has only one change $+\ -$ in the second and third terms; likewise $x^3-0x^2-6x-2=0$ has only one change $+\ -$ in the first and second terms; therefore only one affirmative root, and the remaining two negative.

If, when one of the terms of an equation between two terms with like signs is zero, and if the term be supplied first with an affirmative and then with a negative sign, so as to form two different equations, then if, on the application of the above rule, the number of affirmative roots are unequal, these roots are impossible; for, by inserting a cipher for the term wanting, its sign may be either $+$ or $-$. Now in case of its being $+$, no change will be produced but what exists in the proposed equation; but by writing the negative sign $-$, two additional changes will necessarily be introduced. Thus, in the equation $x^3+qx+r=0$, by introducing $+0$, it becomes $x^3+0+qx+r=0$, in which there is no change; but by introducing -0, it becomes $x^3-0+qx+r=0$, from which we have the two changes $+-$, $-+$. The proposed equation has, in this case, at least two impossible roots.

And if two succeeding terms be wanting, the equation must at least have two impossible roots.

In any equation containing both affirmative and negative roots, it will be most convenient to apply the new method

of extraction to finding the affirmative root, though the same process applies equally to both.

Here I cannot help observing the singular coincidence of the rule which I have given in my Combinatorial Essays for the decomposition of algebraic products with the non-figurate method of extracting the roots of equations.

$$
\begin{array}{l|lll|lll|l}
{}_1A=1 & {}_2B= & & {}_1Aa' & {}_3C= & & Bb'' & \&c. \\
{}_2A=1 & {}_3B= & {}_2B+ & {}_2Ab' & {}_4C= & {}_3C+ & {}_3Bc'' & \&c. \\
{}_3A=1 & {}_4B= & {}_3B+ & {}_3Ac' & \ldots & \ldots & \ldots & \\
{}_4A=1 & \ldots & \ldots & \ldots & {}_nC={}_{n-1}C+{}_{n-1}Bl'' & & & \\
\ldots & {}_nB={}_{n-1}B+{}_{n-1}Al' & & & & & & \\
{}_nA=1 & & & & & & &
\end{array}
$$

The preceding formula exhibits the rule, and simply directs, that *any value is to be found by multiplying the opposite value on the left by its corresponding difference; then adding the product to the number above that which is required.*

Instead of dwelling upon the particulars of the rule in finding the differences a', b', c', &c.; b'', c'', &c.; d''', &c. I shall explain the whole process by an example.

Let it be required to decompose the algebraic product $(x+1)(x+3)(x+6)$ in the form $x^{3|1}+Bx^{2|1}+Cx+D$.

Here, in the form required, 0 is added to x in the first factor, 1 to x in the second, and 2 to x in the third.

To find the differences, subtract 0, 1, 2 respectively from the second parts 1, 3, 6 of the binomial factors of the given product, and the remainders are $a'=1$, $b'=2$, $c'=4$. Again, subtract 0, 1, respectively from 3, 6, and the remainders are $b''=3$, $c''=5$; lastly, subtract 0 from 6, and the remainder is $c'''=6$.

Then applying the preceding formula, we obtain the following arithmetical operation, where it must be observed that

the column $_1A$, $_2A$, $_3A$, &c. is omitted, each term being equal to unity.

Here the values of $_2B$, $_3B$, $_4B$, are found from the three first remainders; thus, $_2B=1$, $_3B={}_2B+2=3$, $_4B={}_3B+4=7$.

$$\begin{array}{l|l|l} _2B=1 & _3C=3 & _4D=18 \\ _3B=3 & _4C=18 & \\ _4B=7 & & \end{array}$$

The values of $_3C$, $_4C$, are found from the second remainders 3, 5; thus, multiply $1={}_2B$ by 3, and the product 3 is the value of $_3C$. Again, multiply 3, the value of $_3B$, by 5, and add the product 15 to $3={}_3C$, and we have $18={}_4C$.

Lastly, multiply 3, the value of $_3C$, by 6, and the product 18 is the value of $_4D$.

Hence the values of $_4B$, $_4C$, $_4D$, are the values of the coefficients B, C, D, in the proposed form,

and $(x+1)(x+3)(x+6)=x^{3|1}+7x^{2|1}+18x+18$;

or, if the above product be required in the form of powers, we must proceed with the numbers 1, 3, 6, instead of the first remainders, the numbers 3, 6 instead of the second remainders, the number 6 instead of the third remainder, in the following form:

$$\begin{array}{l|l|l} 1 & 3 & 6 \\ 3 & 6 & \\ 6 & & \end{array} \qquad \begin{array}{l|l|l} _2B=1 & _3C=3 & _4D=18 \\ _3B=4 & _4C=27 & \\ _4B=10 & & \end{array}$$

Whence $(x+1)(x+3)(x+6)=x^3+10x^2+27x+18$.

As the knowledge of figurate numbers is indispensable in the theory of extracting the roots of equations, and as they have nowhere been treated in so satisfactory a manner as could be wished for, I have fully and clearly demonstrated their most useful properties.

I have also added an article on Arithmetical Equivalents, which will be found exceedingly useful in extracting the roots of numerical equations, where any of the signs of the terms are negative; for, instead of being continually interrupted by subtractions, the whole may be performed by addition.

London,
May 1st, 1820.

MR. P. NICHOLSON having devoted much of his attention to the Mathematical Sciences, takes this opportunity to acquaint his Friends and the Public, that he continues to teach Euclid's Elements, Conic Sections, Algebra, Fluxions, *&c., and their Practical Application to* Trigonometry (Plane and Spherical), Mensuration of Heights and Distances, and of Superfices and Solids.

He likewise teaches Mechanical Drawing, Projection, Projection of the Sphere, Perspective, *&c.*

Letters may be addressed to him at his Residence, No. 39, *Gower Place, Euston Square; or to No.* 3, *Chapel Court, Swallow Street.*

INVOLUTION.

Part I.

Definition.

INVOLUTION is the method of finding the value of a function expressed in one or more affected powers* of an unknown quantity, according to any given value assigned to that unknown quantity†.

The present article of Involution is not given with a view so much to facilitate the method of finding the value of a function expressed in the powers of an unknown quantity in given terms of that quantity, as to familiarize the principle before it is applied to the inverse method, or extraction of roots; for it is in its application to Evolution that it has a decided advantage over any other method hitherto discovered, as will be clearly seen when I come to treat of the second part of this very interesting subject.

Notation.

The reader will frequently find expressions of the form $m^{n|1}$, $m^{n-1|1}$, $m^{n-2|1}$, &c.; these mean respectively

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* By affected powers is here meant such powers as are connected with a numeral factor.

† This is the sense in which Involution is here understood. The definition now given, though very general, is not universal, unless that the following was also included.

Involution is also the method of finding the coefficients and absolute number of a function, in which the various powers of an unknown quantity enter, so that the unknown root, or first power, may have one or more given values; as if it were required to find P, Q, R, &c. and the absolute number N, so that x in the general equation $Px^n + Qx^{n-1} + Rx^{n-2} + \ldots + Zx = N$ may be equal to any one of the n quantities a, b, c, &c.

$$m \times (m+1) \times (m+2) \ldots (m+n-1),$$
$$m \times (m+1) \times (m+2) \ldots (m+n-2),$$
$$m \times (m+1) \times (m+2) \ldots (m+n-3),$$
&c.

So that $m^{n|1}$ means a product consisting of n factors, of which the first is m, and each succeeding factor is one more than the preceding, in the same manner that $m^n = m^{n|0}$ means a product of n factors, each equal to m.

He will also meet with $m^{n|\overline{1}}$, $m^{n-1|\overline{1}}$, $m^{n-2|\overline{1}}$, &c., which are respectively equivalent to the following forms:

$$m \times (m-1) \times (m-2) \ldots (m-n+1),$$
$$m \times (m-1) \times (m-2) \ldots (m-n+2),$$
$$m \times (m-1) \times (m-2) \ldots (m-n+3), \text{ \&c.}$$
&c.

Likewise $1^{n|1}$; $2^{n-1|1}$; $3^{n-2|1}$, &c. respectively imply $1 \times 2 \times 3 \ldots n$; $2 \times 3 \times 4 \ldots n$; $3 \times 4 \times 5 \ldots n$.

The reason of introducing this form of notation for products of which the factors are in arithmetical progression, is to obtain several terms in one line, whereby the law in which they succeed each other is rendered clear to the eye, which conveys the idea at once to the understanding.

When several quantities are placed above a line or between two lines, as in the margin, and a quantity below the line, that quantity stands for the sum of all the quantities above the line, or between the two lines.

$$\begin{array}{l} Av \\ Bav \\ Ca^2v \\ \text{\&c.} \\ \hline Pv \end{array} \qquad \begin{array}{l} \hline pAa \\ qBa^2 \\ rCa^3 \\ \text{\&c.} \\ \hline Q \end{array}$$

The mark thus .— is the sign of subtraction: its use is to direct the sign or signs of the quantities following it, whether + or —, to be changed.

PROPOSITION I.

PROBLEM.

Let $z = a + b + c + \ldots + l$; *to find the value* N *of the function* $Az^n + Bz^{n-1} + Cz^{n-2} + \ldots + Lz$, *in terms of the given quantities* A, B, C, *&c. and* a, b, c, *&c.*

Let $z = v + a$, and let u be the exponent of any power of z; then will

$$z^u = (v+a)^u = v^u + \frac{u^{1|\overline{1}}}{1^{1|1}} a v^{u-1} + \frac{u^{2|\overline{1}}}{1^{2|1}} a^2 v^{u-2} + \frac{u^{3|\overline{1}}}{1^{3|1}} a^3 v^{u-3} + \&c.$$

which, by reversing the factors in each term, will become

$$z^u = 1 v^u + \frac{u^{1|1}}{1^{1|1}} a v^{u-1} + \frac{(u-1)^{2|1}}{1^{2|1}} a^2 v^{u-2} + \frac{(u-2)^{3|1}}{1^{3|1}} a^2 v^{u-3} + \&c.$$

And this value of z^u will again, by the properties of figurate numbers*, become

$$z^u = \frac{1^{u|1}}{1^{u|1}} v^u + \frac{2^{u-1|1}}{1^{u-1|1}} a v^{u-1} + \frac{3^{u-2|1}}{1^{u-2|1}} a^2 v^{u-2} + \frac{4^{u-3|1}}{1^{u-3|1}} a^3 v^{u-3} + \&c.$$

Then, by substituting n, $n-1$, $n-2$, &c. for u, and arranging the values according to the powers of v, we have

$$Az^n = \frac{1^{n|1}}{1^{n|1}} Av^n + \frac{2^{n-1|1}}{1^{n-1|1}} Aav^{n-1} + \frac{3^{n-2|1}}{1^{n-2|1}} Aa^2v^{n-2} + \frac{4^{n-3|1}}{1^{n-3|1}} Aa^3v^{n-3} + \&c.$$

$$Bz^{n-1} = \quad . \quad . \quad \frac{1^{n-1|1}}{1^{n-1|1}} Bv^{n-1} + \frac{2^{n-2|1}}{1^{n-2|1}} Bav^{n-2} + \frac{3^{n-3|1}}{1^{n-3|1}} Ba^2v^{n-3} + \&c.$$

$$Cz^{n-2} = \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \frac{1^{n-2|1}}{1^{n-2|1}} Cv^{n-2} + \frac{2^{n-3|1}}{1^{n-3|1}} Cav^{n-3} + \&c.$$

$$Dz^{n-3} = \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \frac{1^{n-3|1}}{1^{n-3|1}} Dv^{n-3} + \&c.$$

&c. &c.

---

* See Figurate Numbers, at the end of Evolution.

But since the original function is equal to the sum of all the columns, let the coefficients of the powers of v be represented respectively by A_2, B_2, $C_2 \dots L_2$; then will

$$A_2 = \frac{1^{n|1}}{1^{n|1}} A, \text{ but, because } \frac{1^{n|1}}{1^{n|1}} = 1, A_2 \text{ is } = A$$

$$B_2 = \frac{1^{n-1|1}}{1^{n-1|1}} B + \frac{2^{n-1|1}}{1^{n-1|1}} Aa$$

$$C_2 = \frac{1^{n-2|1}}{1^{n-2|1}} C + \frac{2^{n-2|1}}{1^{n-2|1}} Ba + \frac{3^{n-2|1}}{1^{n-2|1}} Aa^2$$

&c. &c. &c. &c. &c.

Now, because the number of terms in the expansion of a binomial exceeds the exponent of its radix by unity, the expansion of the first power will have two terms; the expansion of the second power, three terms; and so on to the expansion of the nth power, which will contain $(n+1)$ terms; therefore both the nth column and the $(n+1)$th column will each contain n terms; and because the last term of the expansion of a binomial has unity for its numeral factor, the terms of the $(n+1)$th column formed by the last terms of the binomial expansions, will each have unity for its numeral factor; and because in each of the preceding expansions the powers of a are increasing from zero in the first term by unity, and the powers of v decreasing by unity to zero in the last; therefore the last column will be divisible by a. Let δ represent $L+Ka+Ja^2+\dots+Aa^{n-1}$, the last column after being divided by a; then will δa represent the last column itself; consequently the first given function $Az^n+Bz^{n-1}+Cz^{n-2}+\dots+Lz$ is transformed to $A_2v^n+B_2v^{n-1}+C_2v^{n-2}+\dots+L_2v+\delta a$. Now, here it may be observed in this transformed quantity, that the part

$A_2v^n+B_2v^{n-1}+C_2v^{n-2}+...+L_2v$, expressed in the powers of v, is exactly the same form as the proposed function expressed in the powers of z. Let the similar expression, consisting of the powers of v, be called the second function, where $z=v+a$, or $v=z-a$. Now, if z be a monomial, or consist of one term a, v will be $=0$; then the original function $Az^n+Bz^{n-1}+Cz^{n-2}+...+Lz=\delta a$.

Suppose, now, that z consists of more terms than one; and because the second function has now the same form as the first, let $v=w+b$; and since $z=v+a$, $z=a+b+w$, or $w=z-a-b$. Substitute $w+b$ in the second function for v, and it will become

$$A_3w^n+B_3w^{n-1}+C_3w^{n-2}+...+L_3w+\delta_2b;$$

then, if z be a binomial, $a+b$ or $z=a+b$, w will be $=0$, or zero; therefore $A_2v^n+B_2v^{n-1}+C_2v^{n-2}+...+L_2=\delta_2b$; whence the original or first function

$$Az^n+Bz^{n-1}+Cz^{n-2}+...+Lz=da+\delta_2b;$$

that is, when $z=a+b$, the first or given function will be transformed to $\delta a+\delta_2b$, which is expressed in terms of a and b, as required. Whence, by the same law, if

$$z=a+b+c+...+l,$$

consisting of p terms, then will

$$Az^n+Bz^{n-1}+Cz^{n-2}+...+Lz=\delta a+\delta_2b+\delta_3c+...+\delta_pl.$$

Then, as the same relation subsists between every two consecutive involutions, the same rule will be repeated for every new figure of the operation; hence the following Rule, which applies to all the steps of the work as they follow in succession.

RULE.

Multiply the first term a in the value of z by the coefficient A of the first term, and place the product on the right hand, and write the second coefficient B above the product in the form of a column, which I shall call B; multiply each member in the column B into a, and place the products on the right-hand, and the third coefficient C above, which will form the column C. Proceed in the same manner till all the coefficients are used; then the sum δa of the last column will be the value of the given function when z is a monomial, or when $z=a$.

But if $z=a+b$, multiply each term of the column B by the corresponding term of the nth order of figurate numbers; each term of column C by each term of the $(n-1)$th order; each term of column D by each corresponding term of the $(n-2)$th order; and so on, through all the columns to the last, which multiply by each term of the second order; then the sum of the products in each column will form the coefficients B_2, C_2, D_2...L_2 of v, the second function. Proceed now in the same manner with b, the second part in the value of z, and the new coefficients B_2, C_2, D_2, as has been done with the first part a in the value of z and the coefficients, B, C, D; then if z consist of two parts $a+b$, the sum of $\delta a+\delta_2 b$ will be the value of the given function; and so on, to as many terms as are in the value of z.

N.B. $A=A_2=A_3$, &c.

EXAMPLE I.

Find the value of the quadratic function x^2+3x, *supposing* $x=5236=5000+200+30+6$.

Here $A=1$, and $B=3$; also $a=5000$, $b=200$, $c=30$, and $d=6$.

Here, in order to place the first denomination in the value of x, I add as many ciphers to the right-hand of the coefficient 3 of the second term, so that the whole number of places may be equal to the whole number of places in the first denomination of x; and multiply 5 by $1=A$, the coefficient of x^2, and the product Ba is 5, which I place under the left-hand cipher; add these two numbers together, and the sum is $\delta=5003$; multiply δ or 5003 by 5, the digit in the first denomination,, and place the product $\delta a=25015$ in any convenient place. Instead of multiplying $B=3$ by 1, and $Aa=5$ by 2, I add δ and the number above it, and the sum 10003 is B_2. I then place the product 2 of the next figure 2 in the value of x, and 1, the coefficient of x^2, one figure to the right of the 5 in the first step; then add as before, and it gives $\delta_2=20406$; multiply δ_2 by $b=2$, it gives $\delta_2 b=20406$, which place one figure to the right of the product δa, and so on; then the sum $27431404=x^2+3x$.

$$
\begin{array}{rl}
0003 & \\
5 & \\
\hline
5003 & =\delta, \\
\hline
10003 & \\
2 & \\
\hline
10203 & =\delta_2, \\
\hline
10403 & \\
3 & \\
\hline
10433 & =\delta_3, \\
\hline
10463 & \\
6 & \\
\hline
10469 & =\delta_4, \\
\hline
\end{array}
\qquad
\begin{array}{rl}
\delta a & =25015 \\
d_2 b & =20406 \\
\delta_3 c & =31299 \\
\delta_4 d & =62814 \\
\hline
x^2+3x & =27431404
\end{array}
$$

Example II.

Find the value of $x^3 - 7035x + 15262754x$, *supposing* $x = 3000+400+50+6 = 3456$ *considered as a tetranomial.*

Here -7035 is equivalent to $2965 - 10000 = \bar{1}2965$, which being used instead of -7035, will save the trouble of subtractions.

	$\bar{}15262754$	9473262
$\bar{1}2965$	$\bar{1}78895$	3995016
3	9	11327520
	$\delta = 3157754$	14692080
	52754	10000730880
1965	7860	
4	16	
	$\delta_2 = 998754$	
	2104754	
3165	15825	
5	25	
	$\delta_3 = 2265504$	
	2428754	
3315	19890	
6	36	
	2448680	

The same by the common method of involution, or substituting 3456 for x :—

$3456 = x$	11943936	15262754
3456	-7035	3456
20736	59719680	91576524
17280	35831808	76313770
13824	83607552	61051016
10368	-84025589760	45788262
$11943936 = x^2$		$Cx = 52748077824$
3456		41278242816
71663616		$x^3 + Cx = 94026320640$
59719680		$-Bx^2 = -84025589760$
47775744		$x^3 - Bx^2 + Cx = 10000730880$
35831808		
$41278242816 = x^3$		

In the regular operation above there are 143 figures; and in the operation by substituting 3456 for x, there are 229 figures.

Scholium.

What has now been shown in finding the value N of the general function $Az^n + Bz^{n-1} + Cz^{n-2} + \ldots + Lz$, in terms of $a + b + c + \ldots + l$ for z, will apply to finding the value of z in the general equation $Az^n + Bz^{n-1} + Cz^{n-2} + \ldots + Lz = N$, without any other demonstration than that which I have given for Involution.

For, suppose we have any means of anticipating a, b, $c \ldots l$, the known portions of z, we have nothing more to do than subtract $\partial a + \partial_2 b + \partial_3 c + \ldots + \partial_p l$ from the absolute number N, or, if they can be found in succession, one after the other, we shall have an operation of the form exhibited in the margin, where observe that ∂a is the greatest product possible, not exceeding N; R_1 is the first remainder; S_1 the figure or figures annexed to the right-hand of R_1, so that R_1, R_2, R_3, &c. are the several remainders, S_1, S_2, S_3, &c., the several periods annexed to R_1, R_2, R_3, &c. It is evident that if N be a surd, there will always be a remainder.

$$
\begin{array}{l}
N\,(a+b+c+\ldots+l \\
.-\partial a \\
\overline{R_1+S_1} \\
\quad .-\partial_2 b \\
\quad \overline{\quad R_2+S_2} \\
\qquad\quad .-\partial_3 c \\
\qquad\quad \overline{\qquad R_3+} \\
\qquad\qquad\quad \&c.
\end{array}
$$

But, as the extraction of roots is a subject of so great importance, I shall treat of Evolution independent of any thing here said of Involution.

EVOLUTION.

Part II.

Definition.

EVOLUTION is the method of extracting the roots of equations of all degrees.

PROPOSITION II.

Theorem.

If in the equation $Az^n + \ldots + Jz^3 + Kz^2 + Lz = N$ the root z be a very small quantity, less than unity, and if a be the nearest digit possible to the root, not exceeding it; and if $a+u$ be substituted in the original equation for z, and the result be put equal to N, then the corresponding coefficients of the new equation will be increased but in a very small degree, and the number generated will be less than the absolute number N.

For, by the binomial theorem,

$$\begin{array}{llll} Lz = L(a+u)^1 = 1La & + 1Lu & & \\ Kz^2 = K(a+u)^2 = 1Ka^2 & + 2Kau & + 1Ku^2 & \\ Jz^3 = J(a+u)^3 = 1Ja^3 & + 3Ja^2u & + 3Jau^2 & + Ju^3 \\ \&c. \quad \&c. & \&c. & & \&c. \end{array}$$

Now, let

$$\begin{aligned} \delta &= 1L + 1Ka + 1Ja^2 + \&c. \\ L_2 &= 1L + 2Ka + 3Ja^2 + \&c. \\ K_2 &= 1K + 3Ja + \&c. \\ J_2 &= 1J + \&c. \\ &\&c. \quad \&c. \end{aligned}$$

Then, in any one of these values the quantity a enters each of the terms in the form of a power, which increases with its distance from the first term in which a does not enter; therefore, as a is supposed to be less than unity, any power of it will be less than itself, and the value of a higher power will be less than a lower power;

$$\text{therefore } \left.\begin{array}{l} \delta = L \\ L_2 = L \\ K_2 = K \\ J_2 = J \end{array}\right\} \text{nearly.}$$
$$\&c. \quad \&c.$$

And because a is, by hypothesis, nearly equal to the root z, but less than z, and since $\delta a = La + Ka^2 + Ja^3 + \&c.$, therefore $La + Ka^2 + Ja^3 + \&c.$ is less than $Lz + Kz^2 + Jz^3 + \&c.$; and consequently δa is less than $Lz + Kz^2 + Jz^3 + \&c.$; and therefore also δa is less than N.

Again, because $L_2 = L + 2Ka + 3Ja^2 + \&c.$, L_2 is greater than L.

For the same reason K_2 is greater than K; J_2 greater than J, and so on. Q.E.D.

Corollary 1.—Hence δ is less than L_2.

Corollary 2.—Hence, if there be three equations, such that, if the second be derived from the first, and the third from the second; then, if the coefficients of the second be nearly equal to the coefficients of the first, the coefficients of the third will be still more nearly equal to those of the second.

Corollary 3.—Hence, in a series of equations derived from each other in the same manner, where the root of the first equation is less than unity, the first or left-hand digits of the corresponding coefficients will, when a certain num-

ber of equations have been obtained, become constant; and the number of constant figures will be augmented by every succeeding equation after that number.

Corollary 4.—Hence, because δ is nearly equal to L_2, but less, if $\delta_1 a$, $\delta_2 b$, $\delta_3 c$, &c. be the series of numbers acquired by each new equation, when the new unknown quantity and the new known part of the root is substituted in the last for its root, the left-hand figures of each number δ_1, δ_2, δ_3, &c. will also, when a certain number of equations have been obtained, become constant; and the number of constant figures will be augmented after this number of equations*.

---

* To illustrate this proposition and its corollaries, let the original equation be $z^3+6z^2+10z=1$; by trial it will be found that .09 is as nearly equal to z as can be expressed by one digit, so as not to exceed it; therefore let $z=.09+u$, which substitute in the given equation; and, proceeding in the same manner from one equation to another, we shall have the following series of equations:—

$$u_1^3+6.27u_1^2+11.1043u_1=1-10.5481\times.09=.050671$$

$$u_2^3+6.282u_2^2+11.154508u_2=.050671-11.129396\times.004=.006153416$$

$$u_3^3+6.2835u_3^2+11.16079075u_3=.006153416-11.15764925\times 0005$$
$$=.000574591375$$

$$u_4^3+6.28365u_4^2+11.1614191075u_4=.000574591375-11.1611049275\times.00005$$

&c. &c. &c.

Whence the disposition of the coefficients, as well as the values of δ_1, δ_2, δ_3, &c. to become constant, is apparent.

PROPOSITION III.

Theorem.

If the signs of an equation be all affirmative, and the absolute number be divided by the coefficient of any term, that root of the quotient which is indicated by the exponent of the unknown power in that term, will be greater than the root of the equation.

For, let $Az^n+Bz^{n-1}+Cz^{n-2}+\ldots+Lz=N$ be the equation, and let Pz^x be any term, of which P is the coefficient and x the exponent of the unknown quantity z.

Then, because the absolute number N is equal, by supposition, to the sum of all the terms, it is greater than any one of them, and consequently greater than Pz^x.

And because when the same operation is performed upon two unequal quantities, whether by addition or subtraction, multiplication or division, involution or evolution, the results will be unequal, and the greater result will be that which arose from the greater quantity;

$$\text{Then, because } N \text{ is} > Pz^x,$$
$$\frac{N}{P} \text{ is} > z^x;$$
$$\text{therefore } \sqrt[x]{\frac{N}{P}} \text{ is} > z. \quad \text{Q.E.D.}$$

PROPOSITION IV.

Problem.

To find the highest denomination in the root of an equation, supposing all the signs affirmative.

If the difference of the coefficients is not very great, and if the coefficient of the single power be greater than the absolute number, divide it by the coefficient of the single power, and the quotient, taking only one signifi-

cant figure, observing its rank or place of decimals, will not be less than the first denomination of the root.

If the absolute number be greater than any of the coefficients, divide it into as many sections as possible, from right to left, containing each as many figures as there are units in the exponent of the highest power; divide the figure or figures that remain on the left-hand by the coefficient of that power; find the nearest corresponding power less than the quotient; to the root of this power annex as many ciphers as the sections not used are in number; then the number thus found may be greater or equal to, but can never be less than, the highest denomination of the root.

But, if any of the intermediate coefficients are exceedingly greater than the rest, proceed in the same manner with this coefficient, divide the absolute number by it, and extract the root, as before.

If the denomination of the root found under any of these cases be too great, it must be reduced till it is found to succeed.

EXAMPLE I.

Find the greatest denomination in the root of the cubic equation $3x^3+5x^2+7x=2$.

Here $2\div7=.2+$, which, being substituted in the equation, will be found to succeed.

EXAMPLE II.

Find the greatest denomination in the root of the equation $3x^3+5x^2+7x=67358429$.

Here the number 67358429, divided into sections according to the exponent 3, is $6\dot{7}35\dot{8}42\dot{9}$; then dividing the left-hand section by 3, the quotient is 22. The nearest cube

to 22 is 8, the root of which is 2; then we may presume that the first denomination of the root does not exceed 200; but, upon trial, it will be found to be the first denomination itself.

EXAMPLE III.

Find the greatest denomination in the root of the equation

$$3x^3+4537x^2+7x=67358429.$$

Here the coefficient, 4537, of x^2 is much greater than the rest; then pointing the absolute number 67358429 according to the exponent 2 of x, we may proceed thus with the operation:

$$4537)\dot{6}7\dot{3}5\dot{8}4\dot{2}9(1$$
$$4537$$

The nearest cube to 1 is 1 itself, and its root is also 1; therefore it may be presumed that the highest denomination of the root is 100, which upon trial is found to succeed.

PROPOSITION V.

THEOREM.

The root of the equation $Az^n+Bz^{n-1}+Cz^{n-2}+...+Lz=N$ may be found either correctly or to any number of places, in the following manner:—First find the nearest digit a to the root, by Problem I; then, if a be not the exact root, let the excess of the root above a be denoted by u; then in the equation for z substitute $u+a$, and it will become of the form

$$A_2u^n+B_2u^{n-1}+C_2u^{n-2}+...L_2u+\delta a=N;$$

and, by transposing δa, it will become

$$A_2u^n+B_2u^{n-1}+C_2u^{n-2}+...+L_2u=N-\delta a=N_2;$$

then the root of this equation will be less than z, the root of the original equation, by the quantity u.

Proceed in the same manner with v as with u, and find b, the nearest digit, or highest denomination; then if, as before, b be not exactly equal to u, let $b+v$ be equal to u, and substitute $b+v$ for u in the last equation, and we shall have a third equation of the form

$$A_3v^n+B_3v^{n-1}+C_3v^{n-2}+\ldots+L_3v+\delta_2b=N_2,$$

and, by transposing δ_2b, it will become

$$A_3v^n+B_3v^{n-1}+C_3v^{n-2}+\ldots+L_3v=N_2-\delta_2b=N_3;$$

then the root v of this equation is less than the root u of the last equation by the digit b, and less than the root of the original equation by the consecutive digits a and b or $a+b$.

Proceed in the same manner from one equation to another, until the root is exactly found, or sufficiently near to answer the purpose*.

Scholium.—But as the progress made in this way will be exceedingly slow, from the trouble of raising the powers from a binomial root, in order to form each new equation, I shall show how the process may be carried on by an arithmetical operation, which is derived from the principle here shown.

---

* To illustrate this, let it be required to find the root of the equation $x^3-2x=5$.

We shall find 2 to be the first denomination of the root; therefore, let $x=2+u$, which, being substituted for x, the equation becomes $u^3+6u^2+10u+4=5$, or, by transposition, $u^3+6u^2+10u=1$; divide 1 by 10, and the quotient is .1, which, upon trial, will be found to be too great; therefore let .09 be tried, and it will be found to succeed; therefore substitute $.09+v$ for u, and we shall have the equation $v^3+6.27v^2+11.1043v+.949329=1$, or, by transposition, $v^3+6.27v^2+11.1043v=1-.949329=.050671$, and so on; therefore, since u is greater than v by .09, and x is greater than u by 2, the root of the original equation is greater than v by 2.09, or $x=2.09+v$.

PROPOSITION VI.

PROBLEM.

To find the root in numbers of the quadratic equation

$$Az^2+Bz=N.$$

Find a, the first known portion of the root, by Prop. IV, and let v be the remaining part; then will $z=v+a$;

$$\text{therefore} \quad Bz = \quad 1Bv + Ba$$
$$Az^2=Av^2+2Aav+Aa^2.$$

Let the sum of the coefficients of v be denoted by B_2, and the given numbers in the third column be denoted by δa; then will the coefficient of v^2 be the same as in the given equation; therefore, by substituting $v+a$ for z, the proposed equation will become

$$Av^2+B_2v+\delta a=N,$$

which, by transposition, will become

$$Av^2+B_2v=N-\delta a.$$

Let $N-\delta a=N_2$, and the result,

$$Av^2+B_2v=N_2,$$

will be the new equation, where the coefficient B_2 and δ are equal to the known values above. And here we must observe, that the coefficient of v, when each term of it is divided by its respective figurate number, is equal to δ. From this circumstance we shall not be under the necessity of using the third column, as δ may be found from the second.

In order to form the rule, the coefficient of v is here exhibited, the sum being called B_2,

$$\begin{array}{c} 1B \\ 2Aa \\ \hline B_2 \end{array}$$

or $B_2=1B+2Aa$, and $\delta=B+Aa$.

But when the work has advanced one or two steps, we shall have no occasion for the coefficients B_2, B_3, B_4, of the succeeding equations, as the last divisor will be sufficient to ascertain the new figure of the root. I shall here show how the divisors may be found without forming the coefficient of the second term; and, instead of the succeeding equations, an arithmetical operation in steps, each repeated in the same manner.

For this purpose, let a, b, c, &c. be the successive figures of the root; then, since

$$\delta = B + Aa$$
$$B_2 = B + 2Aa.$$

Again, $\delta_2 = B_2 + Ab = B + 2Aa + Ab = \delta + A(a+b)$

$$B_3 = B_2 + 2Ab.$$

Again, $\delta_3 = B_3 + AC = B_2 + 2Ab + Ac = \delta_2 + A(b+c)$

$$B_4 = B_3 + 2Ac$$
$$\delta_4 = B_4 + Ad = B_3 + 2Ac + Ad = \delta_3 + A(c+d)$$

&c. &c. &c. &c.

Therefore the divisor of the new step is equal to the divisor of the preceding step plus the product of the coefficient of the highest power multiplied into the number formed of the sum of the last and present figures.

Rule.

Find the first figure or part a of the root; under the coefficient B of the single power set the product of the coefficient A of the highest power, and the figure now ascertained; add these two numbers together, and place the sum underneath for a divisor.

On the right of the divisor draw a line; and on the right of this line, and in a line with the divisor, write the absolute number, and draw another line on the right of the absolute number; on the right of this line, and in the same line with the divisor and absolute number, place the first figure of the root: multiply the divisor by the first figure of the root, and subtract the product from the absolute number.

If there is no remainder the work is done; but if there is, suppose a cipher annexed to the remainder; enquire how often the divisor is contained in the number thus increased with the cipher; put the number of times in the second place of the root; multiply the two figures of the root by A, the coefficient of the highest power, and add the product to the last divisor, and place the sum underneath for the new divisor.

Proceed in the same manner with the divisor now formed as before, and so on from step to step, observing in extracting the roots of pure numbers, and in the decimal part of a quadratic equation in forming the new divisor, to set the product of the last two figures by A one place to the right of the number above; and, in this case, add two ciphers to each remainder, for a new resolvend or absolute number.

EXAMPLE I.

Extract the root of the quadratic equation $3x^2+7x=5$.

Divide 5 by 7, and the quotient .7 ought, by the rule, to be the first figure of the root: the shortest method of trying whether it is or not, is by the operation itself; thus,

```
 7
 2.1
 ----
 9.1)5  (.7
     637
     ---
        remainder negative.
```

.7 is therefore too great; and by proceeding in the same manner 6 will be found to be too great; I shall, therefore, take .5, which will now be found to succeed; whence

```
 7
 1.5                 (3)
 ---
 8.5   |5       (.5733844
  171  |425
 -------------
 1021     |7500
  219     |7147
 ----------------
 10429    |35300
    99    |31287
 ----------------
 104389   |401300
    114   |313167
 ------------------
 1044004   |8813300
     252   |8352032
 ---------------------
 10440292   |46126800
      132   |41761168
 ----------------------
 104403052  |436563200
            |417612908
            -----------
              18950992
```

To anticipate the next figure of the root, divide 750 by 85, and the quotient is 8; therefore by proceeding to complete the step, it will be found to be too much. This step of the work must, therefore, be effaced. Try 7; then multiply 57 by 3, and place the product, 171, under 85, one figure to the right of the unit's place; add these two numbers together, and multiply the sum 1021, which is the divisor, by the new figure 7, and subtract the product, 7147, from 7500, and the remainder is 353.

Again, suppose a cipher annexed to 353, and divide 3530 by 1021, and the quotient is 3, which will be found to succeed; place 3 in the root, multiply 73 by 3, and add the product, 219, to 10210, and the sum, 10429, is the new divisor. Proceed from step to step in the same manner.

EXAMPLE II.

Extract the root of the quadratic equation $x^2+9x=5$.

OPERATION.

In addition to the explanation accompanying the last Example, it may be farther observed here, that, in the commencement of this operation, the first figure of the root 5 is found at once to be the true figure: as to the succeeding figures, every one will be found correct without trial; and as the figures of the divisors become more constant, as many figures of the root may be found by plain division as there are constant figures in the last divisor.

```
9.
 .5                   (1)
9.5 | 5         (.524937
 52 | 475
1002     | 2500
     24  | 2004
10044     | 49600
       49 | 40176
100489     | 942400
        93 | 904401
1004983     | 3799900
         37 | 3014949
10049867     78495100
             70349069
              8146031
```

Thus, the next figure of the root of this Example would be 8: multiply 78 by 1, and set the product under the last divisor, omitting the last figure 8 of the product; then divide the remainder, 81406031, by the divisor 10049874, by contracted division; thus,

```
1004987|4)8146031(810561
          8039899
           106132
           100498
             5634
             5024
              610
              600
               10
               10
```

Therefore the root is

$$x=.524937810561,$$

which, if the work is right, will be correct to the last place of figures.

EXAMPLE III.

Extract the square root of the number 219654328575.

```
4                    (1)
4   |219654328576(468672
46  |16 · · · · ·
86   |596
 68  |516
928   |8054
  86  |7424
9366   |63053
   67  |56196
93727   |683685
   72   |656089
937342   |2759676
         |1874684
           884992
```

Here, since the square of any single digit can never exceed two places of figures, and since in the square of any number, with ciphers annexed, the number of ciphers in the product or square are always double the number of ciphers in the root, we may omit the ciphers altogether by pointing the given number off in couplets; then the part remaining on the left, whether it consists of one or two figures, will be the resolvend, from which the nearest square is to be taken.

This operation of extracting the root of a quadratic equation, or the square root, differs very little from the common process of extracting the square root in the appearance of the operations; but the common method of extracting the square root will not apply to extracting the root of a quadratic equation, except in the case of its being a pure power.

The method of proving the work of extracting the square root, is simply to square the root and add the remainder to the square.

To prove a quadratic equation, square the root, multiply the root by its coefficient, and the square by its coefficient; then add the two products and the remainder together, and the sum, if the work is right, will be the absolute number.

PROBLEM.

To find the nth *root in numbers of the general numerical equation* $Az^n+Bz^{n-1}+Cz^{n-2}+\ldots+Lz=N$.

Find a the first known portion of the root, and let v be the remaining part; then will $z=v+a$; also let u be any power of z; therefore, by the binomial theorem,

$$z^u=(v+a)^u=v^u+\frac{u^{1|\overline{1}}}{1^{1|1}}av^{u-1}+\frac{u^{2|\overline{1}}}{1^{2|1}}a^2v^{u-2}+\frac{u^{3|\overline{1}}}{1^{3|1}}a^3v^{u-3}+\&c.$$

But, by reversing the order of the factors,

$$z^u=v^u+\frac{u^{1|1}}{1^{1|1}}av^{u-1}+\frac{(u-1)^{2|1}}{1^{2|1}}a^2v^{u-2}+\frac{(u-2)^{3|1}}{1^{3|1}}a^3v^{n-3}+\&c.;$$

and this equation will be equivalent to the following†:

$$z^u=\frac{1^{u|1}}{1^{u|1}}v^u+\frac{2^{u-1|1}}{1^{u-1|1}}av^{u-1}+\frac{3^{u-2|1}}{1^{u-2|1}}a^2v^{u-2}+\frac{4^{u-3|1}}{1^{u-3|1}}a^3v^{u-3}+\&c.$$

Then, by substituting n, $n-1$, $n-2$, &c. for u, and multiplying the respective powers and their values by the coefficients, we have the following value of each of the terms of the given equation; viz.

$$\begin{array}{lllll}
Az^n = & \frac{1^{n|1}}{1^{n|1}}Av^n & +\frac{2^{n-1|1}}{1^{n-1|1}}Aav^{n-1} & +\frac{3^{n-2|1}}{1^{n-2|1}}Aa^2v^{n-2} & +\frac{4^{n-3|1}}{1^{n-3|1}}Aa^3v^{n-3}+\&c. \\
Bz^{n-1}= & \ldots & \frac{1^{n-1|1}}{1^{n-1|1}}Bv^{n-1} & +\frac{2^{n-2|1}}{1^{n-2|1}}Bav^{n-2} & +\frac{3^{n-3|1}}{1^{n-3|1}}Ba^2v^{n-3}+\&c. \\
Cz^{n-2}= & \ldots & \ldots & +\frac{1^{n-2|1}}{1^{n-2|1}}Cv^{n-2} & +\frac{2^{n-3|1}}{1^{n-3|1}}Cav^{n-3}+\&c. \\
Dz^{n-3}= & \ldots & \ldots & \ldots & +\frac{1^{n-3|1}}{1^{n-3|1}}Dv^{n-3}\ +\&c. \\
\&c. & & & & \&c. \qquad \&c.
\end{array}$$

Then let A_2, B_2, C_2, &c. be the coefficients of the new equation; then their values will be the sum of the known parts in each of the columns of the values of the given equation; viz.

$$A_2=\frac{1^{n|1}}{1^{n|1}}A=A$$

$$B_2=\frac{1^{n-1|1}}{1^{n-1|1}}B+\frac{2^{n-1|1}}{1^{n-1|1}}Aa$$

$$C_2=\frac{1^{n-2|1}}{1^{n-2|1}}C+\frac{2^{n-2|1}}{1^{n-2|1}}Ba+\frac{3^{n-2|1}}{1^{n-2|1}}Aa^2$$

$$D_2=\frac{1^{n-3|1}}{1^{n-3|1}}D+\frac{2^{n-3|1}}{1^{n-3|1}}Ca+\frac{3^{n-3|1}}{1^{n-3|1}}Ba^2+\frac{4^{n-3|1}}{1^{n-3|1}}Aa^3.$$

&c. &c. &c.

But as the number of terms in a binomial is one more than the number of units contained in the exponent of the power; therefore the number of columns is one more than n, the number of terms in the original equation; and as the second part a of a binomial raised to any power will always be a factor or power in the last term of its value, the $n(+1)$th column will be divisible by a; therefore let δ represent the sum of the remaining factors; then will the value of δ be

$$\delta=\frac{1^{n-n|1}}{1^{n-n|1}}L+\frac{2^{n-n|1}}{1^{n-n|1}}Ka+\frac{3^{n-n|1}}{1^{n-n|1}}Ja^2+\text{\&c.}; \text{ that is,}$$

$$\delta=L+Ka+Ja^2+\ldots+Aa^{n-1};$$

therefore the values of all the terms of the new equation are known; consequently the new quantity, of which N is the value, is

$$Av^n+B_2v^{n-1}+C_2v^{n-2}+\ldots+L_2v+\delta a=N;$$

that is, by transposition,

$$Av^n+B_2v^{n-1}+C_2v^{n-2}+\ldots+L_2v=N-\delta a;$$

that is, putting $N_2=N-\delta a=$ the remainder, the new equation will be

$$Av^n+B_2v^{n-1}+C_2v^{n-2}+\ldots+L_2=N_2.$$

D

The general form of one complete step of the arithmetical operation for extracting the roots of equations of all degrees in numbers is as follows; but it will be convenient to invert the columns, and consequently the order of the coefficients and figurate numbers, for the reason hereafter explained:

$$
\begin{array}{c|c|c|c|c|l}
 & & & & & 1L \\
 & & & \frac{1^{n-3|1}}{1^{n-3|1}}D & \&c.\ \&c.\ \&c. & 2Ka \\
 & & & & & 3Ja^2 \\
 & & \frac{1^{n-2|1}}{1^{n-2|1}}C & \frac{2^{n-3|1}}{1^{n-3|1}}Ca & \&c.\ \&c.\ \&c. & \vdots \\
 & \frac{1^{n-1|1}}{1^{n-1|1}}B & \frac{2^{n-2|1}}{1^{n-2|1}}Ba & \frac{3^{n-3|1}}{1^{n-3|1}}Ba^2 & \&c.\ \&c.\ \&c. & \vdots \\
A & \frac{2^{n-1|1}}{1^{n-1|1}}Aa & \frac{3^{n-2|1}}{1^{n-2|1}}Aa^2 & \frac{4^{n-3|1}}{1^{n-3|1}}Aa^3 & \&c.\ \&c.\ \&c. & nAa^{n-1} \\
\hline
A_2 & B_2 & C_2 & D_2 & \&c.\ \&c.\ \&c. & L_2
\end{array}
$$

And because the last column but one has the literal parts of its terms respectively expressed in the same manner, by leaving out the figurates as the last term itself, when divided by a, the first part of the root, the value of δ, or the divisor, may be found by the nth column, by leaving or dividing out the figurates.

Here, as the first term of each order of figurate numbers is unity, the number at the head of each column is the respective coefficient of the original equation, as it comes in order.

To apply this tabular rule to a cubic equation, let $n=3$; therefore the above general table becomes

$$
\begin{array}{r|l}
 & 1C \\
1B & 2Ba \\
3Aa & 3Aa^2 \\
\hline
B_2 & C_2
\end{array}
$$

General Rule.

In any present step, find the portion a_n of the root for that step; place the first coefficient (A) above the root, if not already done; multiply the first coefficient by the anticipated part (a_n) of the root; and put the product in any convenient place, and the second coefficient above the product. Multiply these two members by the anticipated part of the root, and place the product opposite, in another column on the right, and the third coefficient above. Proceed in the same manner, until all the n coefficients have been brought down; then the sum of all the terms in the last column is the divisor. Multiply the divisor by the anticipated figure of the root; then, if the present step be the first step of the work, subtract the product from the absolute number; but if it is any other step, and if the anticipated portion of the root belongs to a whole number, annex a cipher to the remainder, and subtract the product from the number thus increased; but if the anticipated portion of the root belongs to a pure power, or to a decimal portion or number, annex as many ciphers as in the exponent of the degree of the root to be extracted; then subtract the product, and in this case observe, that in every two consecutive numbers, in any column or class, to place the lower number one figure to the right of the number above. If the remainder has a contrary sign to the resolvend, the anticipated figure has been taken too great, and the work must be repeated; and if there is no remainder, the work is done.

But if there is a remainder, we must find the new coefficients, and proceed as above. To obtain these, multiply each term of the first column respectively by each of the two first terms of the nth order of figurate numbers; each of the three terms of the second column respectively by each of the three first of the $(n-1)$th order of figurates, and so

on; then the sum of the products will be the coefficient of the same number in the new equation, as there are terms in the column or class.

This operation is so uniform, that it can hardly be misunderstood.

N.B. The last coefficient is the trial divisor.

The following table exhibits the form of the arithmetical operation: the figurates, being understood, are left out, and placed over their respective columns:

1, 3	1, 2, 3
	C
B	Ba
Aa	Aa^2
	δ
	C_2
B_2	$B_2 b$
Ab	Ab^2
	d_2
	C_3
B_3	$B_3 c$
Ac	Ac^2
	δ_3
&c.	&c.

(A)

$$
\begin{array}{l}
\delta)\,N\,(a+b+c+\&c. \\
\quad\ \ \delta a \\
\hline
\delta_2)\ R_1+P_1 \\
\qquad\quad \delta_2 b \\
\hline
\delta_3)\,R_2+P_2 \\
\qquad\quad \delta_3 c \\
\hline
\qquad\quad R_3 \\
\qquad\quad \&c.
\end{array}
$$

R_1 being that portion of the absolute number on the left-hand from which δa is subtracted; P_1, P_2, &c., the figures annexed to the respective remainders R_1, R_2, R_3, &c., which will be one or three figures, according as the new figure belongs to whole numbers or decimals.

If the arithmetician observes the rules for placing whole numbers and decimals, he cannot find any difficulty; however, I shall remark, that when the figure now obtaining is a decimal in the three numbers which form the divisor δ_{n-1}, and the value of C_n, as also in the two numbers which form the second coefficient, any lower number must stand one figure to the right of that above it.

EXAMPLE IV.

Find the root of $x^3-2x=5$.

Here the value of the greatest denomination in the root is 2.

1,3	1,2,3
	−2
0............	0
2............	4
	2
	10
6	54
09	81
	105481
	111043
627	2508
4	16
	11129396
	11154508
6282	31410
5	25
	1115764925
	1116079075
62835	314175
5	25
	111611049275
	111614191075
628365	628365
1	1
	11161425391151
	11161431674803
6283653	25134612
4	16
	1116143418826436
	1116143670172588
62836542	502692336
8	64
	111614372|044182224

(1)
5(2.09455148
4
1000000
949329
50671000
44517584
.6153416000
5578824625
.574591375000
558055246375
16536128625000
11161425391151
5374703233849000
4464573675305744
910129558543256000
892914976353457792
17214582189798208

Whence we shall find $x=2.09455148$.

But by the following supplement the number of figures in the whole root may in general be tripled.

For this purpose we must find the new coefficients 62836544 and 1116143770711105712, and proceed according to the principles already laid down, observing, in general, to cut off one figure from the class of numbers on the right-hand by a vertical line; two from the next class on the left; three from the next; four from the next; and so on, to the second coefficient inclusive, in each step of the operation; but if the figure of the root has a cipher before it, cut off two figures from the right-hand of the right-hand class; four figures from the right-hand of the next class on the left; six from the right-hand of the third class; and so on; but in the present example of a cubic equation there are only two classes to be attended to.

To secure as many true figures as possible in the root, it will be advisable to set down all the figures cut off from the last; so that when these figures are multiplied by the figure of the root upon trial, the proper figure to be carried to the product on the left-hand of the line may be exactly discovered; and thus the whole product may be carried, without any loss, to the next class on the right. The vertical line drawn in each class will regulate the corresponding vertical denominations, as may be seen in the following process for contracting the work :—

```
           |11161437707110571|2
6283654|44 |          6283654|4
           |11161437713394225|6
           |11161437719677880|0
62836|54   |           314182|7
           |11161437722819707|7
           |1116143772596153
628|36     |            2513|4
           |1116143772621287|7
           |111614377264642|2
6|28       |             12|5
           |111614377264767|7
           |11161437726489|9

                          17214582189798208(1
                          11161437713394225
                           60531444764039983(5
                           55807188614098853
                            4724256149941300(4
                            4464575090485l4
                             25968105945616(2
                             22322875452953
           1116143772648)3645230492663(3
                              3348431317946
                               296799174717(2
                               223228754529
                                73570420188(6
                                66968626358
                                 6601793836(5
                                 5580718863
                                 1021074967(9
                                 1004529395
                                  16545372(1
                                  11161437
                                   5384135(4
                                   4464575
                                    919560(8
                                    892915
                                     26645(2
                                     22322
                                      4323(3
                                      3348
                                       975(8
                                       892
                                        83(7
                                        77
                                         6
```

Whence $x=2.09455148154232659148 2387$.

This number has been proved to be true to the 23d decimal place, by substituting the root in the original equation. The proof required 2161 figures; whereas the regular extraction and this supplement requires only 965 together.

Example V.

Find the root of $x^4+5x^3+4x^2+3x=105$.

By first finding 4 figures in the root, we may then find the new coefficients, &c use the contracted operation as below; and by this we shall treble the number of figures in the whole root.

1,4	1,3,6	1,2,3,4
		3
	4	8
5	10	20
2	4	8
		39
		111
	58	116
13	26	52
2	4	8
		123128
		135792
	6604	6604
138	138	138
1	1	1
		136453781
		137116944
	664546	4651822
1384	9688	67816
7	49	343
		137582804703

(1)
105(2·217
78
270000
246256
237440000
136453781
1009862190000
963079632921
··46782557079

Whence $x=2{\cdot}217$.

				13804934425	2
		667455	34	2002366	0
13	868	41	60	124	8
				13806936916	
				1380893953	1
	013	6675	80	20027	4
				1380913980	5
				138093400	7
		66	75	534	0
				138093934	7
				13809446	8
			66	5	3
				13809452	1
				1380945	7

	46782557079(3
	41420810748
	5361746331(3
	4142741940
	1219004391(8
	1104751472
	114252919(8
	110475616
1380945)	3777303(2
	2761890
	1015413(7
	966661
	48752(3
	41428
	7324(5
	1904
	420(3
	414
	6

Whence $x=2{\cdot}217338827353$.

From what has been done, I will show how to perform the operation without the figurate factors. For this purpose I shall again exhibit the values of the new coefficients; and here I must remind the Reader, that the exponents of the figurates $n-1$, $n-2$, $n-3$, &c. respectively denote the nth, the $(n-1)$th, the $(n-2)$th, &c. order of figurate numbers. —*See* the article on Figurate Numbers at the end.

The coefficients, with their values, are as follow:

$$A_2 = A$$

$$B_2 = B + \frac{2^{n-1|1}}{1^{n-1|1}} Aa$$

$$C_2 = C + \frac{2^{n-2|1}}{1^{n-2|1}} Ba + \frac{3^{n-2|1}}{1^{n-2|1}} Aa^2$$

$$D_2 = D + \frac{2^{n-3|1}}{1^{n-3|1}} Ca + \frac{3^{n-3|1}}{1^{n-3|1}} Ba^2 + \frac{4^{n-3|1}}{1^{n-3|1}} Aa^3$$

. . . .

$$L_2 = L + 2Ka + 3Ja^2 + 4Ia^3 + \ldots + nAa^{n-1}$$

Then, because in figurate numbers the sum of x terms of any vertical column v is equal to the $(v+1)$th term of the xth order, therefore in every order of figurate numbers, when the first term is taken away, the remaining terms may be decomposed into as many of the first consecutive orders of figurate numbers as there are units in the exponent of the order to be decomposed; but, since the exponent of the order of figurate numbers, in the value of any coefficient, is one more than the power of the unknown quantity belonging to that coefficient, therefore, taking away B in the value of B_2, the remaining term may be resolved into n orders, consisting of one term each; taking away C in the value of C_2, the remaining two terms may be resolved into $n-1$ orders, consisting of two terms each; and so on.

Therefore, suppose each coefficient to be decomposed; and let

$$\begin{array}{ll} B_2 \text{ be represented by} & B + {}_1B + {}_2B + \dots + {}_nB \\ C_2 & C + {}_1C + {}_2C + \dots + {}_{n-1}C \\ D_2 & D + {}_1D + {}_2D + \dots + {}_{n-2}D \\ \vdots \quad \dots & \dots \\ L_2 & L + {}_1L + {}_2L. \end{array}$$

Then, by actual decomposition of the values of these coefficients, we have

$$A_2 = A$$

$$B_2 = B + \left| \begin{array}{l} Aa = {}_1B, \text{ the first order} \\ Aa = {}_2B, \text{ the second order} \\ Aa = {}_3B, \text{ the third order} \\ \dots\dots\dots \\ Aa = {}_nB, \text{ the } n\text{th order} \end{array} \right.$$

$$C_2 = C + \left| \begin{array}{lcl} 1Ba + 1Aa^2 & = & {}_1C, \text{ the first order} \\ 1Ba + 2Aa^2 & = & {}_2C, \text{ the second order} \\ 1Ba + 3Aa^2 & = & {}_3C, \text{ the third order} \\ \dots\dots\dots & & \\ 1Ba + (n-1)Aa^2 & = & {}_{n-1}C, \text{ the } (n-1)\text{th order} \end{array} \right.$$

$$D_2 = D + \left| \begin{array}{lcl} 1Ca + 1Ba^2 + 1Aa^3 & = & {}_1D, \text{ the 1st ord.} \\ 1Ca + 2Ba^2 + 3Aa^3 & = & {}_2D, \text{ the 2d ord.} \\ 1Ca + 3Ba^2 + 6Aa^3 & = & {}_3D, \text{ the 3d ord.} \\ \dots\dots\dots & & \\ 1Ca + \frac{2^{n-3|1}}{1^{n-3|1}}Ba^2 + \frac{3^{n-3|1}}{1^{n-3|1}}Aa^3 & = & {}_{n-2}D, \text{ the } (n-2)\text{th ord.} \end{array} \right.$$

&c. &c. &c. &c.

Let V_2 and W_2 represent any two new consecutive coefficients, corresponding to V and W in the original equation; then in the value of the new leading coefficient V_2, let the sum of V, the corresponding coefficient of the given equation, and the first order ${}_1V$, when multiplied into a, be called a product of the first order; and let any other order ${}_uV$, when multiplied into the part a of the root, be called a product of the order u.

It is evident, from the decomposed values now exhibited, that ${}_1W=(V+{}_1V)a$; and because, in any two consecutive orders of figurate numbers, if any term of the following order be added to the following term of the leading order, the sum will be the corresponding term of the following order; that is, equal to the term following the term first mentioned of the following order. Therefore

$$ {}_{u+1}W={}_uW+{}_{u+1}Va. $$

From these two properties we shall have the following table:

A	$B=B$	$C=\quad C$	&c.	$L=\ L$
0	${}_1B=Aa$	${}_1C=\ (B+\ {}_1B)a$	&c.	${}_1L=(K+{}_1K)a$
0	${}_2B=Aa$	${}_2C=\ {}_1C+\ {}_2Ba$	&c.	${}_2L={}_1L+{}_2Ka$
0	${}_3B=Aa$	${}_3C=\ {}_2C+\ {}_3Ba$		$L_2=L_2$
0	${}_4B=Aa$	. . . .		
0	. . . .	. . . .		
.	. . . .	. . . .		
.	. . . .	${}_{n-1}C={}_{n-2}C+{}_{n-1}Ba$		
.	${}_nB=Aa$	$C_2=\qquad C_2$		
A_2	$B_2=B_2$			

The preceding table exhibits the rule for extracting the root of any equation, to whatever degree it may rise; it also shows how one equation may be transformed to another: but the following disposition will be more convenient for practice in the extraction of roots.

$$
\begin{array}{c|c|c|c|c}
A & B = B & C = C & \&c. & L = L \\
0 & {}_1B = Aa & {}_1C = Pa & \&c. & {}_1L = Za \\
\hline
 & P & Q & \&c. & \delta = \delta \\
0 & {}_2B = Aa & {}_2C = {}_1C + {}_2Ba & \&c. & {}_2L = {}_1L + {}_2Ka \\
\cline{5-5}
0 & {}_3B = Aa & {}_3C = {}_2C + {}_3Ba & \cdot\ \cdot & L_2 = L_2 \\
0 & {}_4B = Aa & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot & \\
\cdot & \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \&c. & \\
\cdot & \cdot\ \cdot\ \cdot\ \cdot & {}_{n-1}C = {}_{n-2}C + {}_{n-1}Ba & & \\
\cline{3-3}
\cdot & {}_nB = Aa & C_2 = C_2 & & \\
\cline{2-2}
 & B_2 = B_2 & & & \\
\hline
A_2 & & & &
\end{array}
$$

then, for the divisor δ_2 of the next step or equation, we have, by the same law, supposing the next figure b ascertained,

$$
\begin{array}{c|c|c|c|c}
A_2 & B_2 = B_2 & C_2 = C_2 & \&c. & L_2 = L_2 \\
0 & {}_1B_2 = Ab & {}_1C_2 = P_2b & \&c. & {}_1L_2 = \gamma_2 b \\
\hline
 & P_2 & Q_2 & \&c. & \delta_2 = \delta_2 \\
 & \&c.\ \ \&c. & \&c.\ \ \&c. & &
\end{array}
$$

But from what I have observed before, that, when the work has advanced to decimals, we may use the divisor instead of the new coefficient of the single power, which has been used as a trial divisor; it will, therefore, be unnecessary to take the sums of the quantities which form the coefficients of the next step by themselves, as the value of δ_2, the next divisor, can be found by the following table, which is equivalent to the last:

A	$B=B$	$C=\ C$	$D=\ D$	&c.
	${}_{1}B=Aa$	${}_{1}C=\ Pa$	${}_{1}D=\ Qa$	
$\bar{A}$	P	Q	R	
	${}_{2}B=Aa$	${}_{2}C=\ {}_{1}C+\ {}_{2}Ba$	${}_{2}D=\ {}_{1}D+\ {}_{2}Ca$	
	${}_{3}B=Aa$	${}_{3}C=\ {}_{2}C+\ {}_{3}Ba$	${}_{3}D=\ {}_{2}D+\ {}_{3}Ca$	&c.
	${}_{4}B=Aa$	${}_{4}C=\ {}_{3}C+\ {}_{4}Ba$	. . . .	
	${}_{5}B=Aa$	. . . .	. . . .	
			${}_{n-2}D={}_{n-3}D+{}_{n-2}Ca$	&c.
	. . . .	. . . .		
	. . . .	${}_{n-1}C={}_{n-2}C+{}_{n-1}Ba$		
	${}_{n}B=Aa$			
	Ab	P_2b	Q_2b	
$\bar{A}$	P_2	Q_2	R_2	&c.
	&c. &c.	&c. &c.	&c. &c.	&c.

The last class will be

$$
\begin{array}{|c}
L=\ L \\
{}_{1}L=\ Za \\
\hline
\delta \\
{}_{2}L={}_{1}L+{}_{2}Ka \\
Z_2b \\
\hline
\delta_2
\end{array}
$$

Here $Aa+Ab=A(a+b)$. *See* the practical operations following, at the bottom of the second class.

We may now observe, that if the first denomination a be given, δ, the first divisor, can be found; but as we can anticipate a near or exact value of a, we may perform the first part of the operation upon trial; and if the product, δa, exceeds the absolute number, a must be reduced. Suppose, then, δa found so as to be equal or less: if δa is equal to the absolute number, the work is finished, which only requires the three first rows.

But if there is a remainder, find the numbers in all the rows of the second step, by the last figure of the root except the first and last, the first being already found in obtaining δ, the first divisor.

Anticipate the second figure of the root by the first divisor, and by it find the last row, and the first row of the next step; then the last number of the first row of the third step will be δ_2; subtract $\delta_2 a$ from the last remainder, if equal or less: if equal to the last remainder, the work is done; but if greater, b must be made less, and if less, the work must be repeated, as before; and so on.

Corollary 1.

Hence the whole number of classes in any step are the same as the number of terms in the proposed equation.

Corollary 2.

Hence the first class of every step is the coefficient of the first term of the proposed equation.

Corollary 3.

Hence the first step of the process contains two numbers in each class except the first, which consists of one only; and the two numbers in any class after the first, stand opposed to the two numbers in the next class on the one or on either side of it.

Corollary 4.

Hence in any step after the first, the second and third classes contain each as many terms or numbers as there are terms in the proposed equation; and any succeeding class, after the third, will have one term less than that which next precedes it.

Definitions.

1. The first, second, third, &c. columns are the first, second, third, &c. columns from the left-hand.

2. The first, second, third, &c. terms or numbers of any column are the first, second, third, &c. numbers from the top of that column.

3. Two corresponding numbers of any two columns are any two terms of the same number.

4. A class in any step is that portion of a column which belongs to that step.

Rule, and illustrative Example.

Let it be required to extract the root of the equation $3x^4+4x^4+5x^3+6x^2+7x=5{\cdot}0400084375$.

1. *Write the coefficients of the given equation in a line, but detached from each other.*

No. 1

3 4 5 6 7

2. *Find a near value to the first figure of the root by Prop. iv, and let this figure be rectified by the operation.*

Here the first figure of the root will be found to be ·4, which will be obtained by the next part of the operation.

3. *Place a cipher under the first coefficient, in order to form the first class. Multiply the sum of the two given terms forming any class by the new figure of the root, and the product will be the second term of the next right-hand class; and the sum of the two terms forming the last class will be the divisor.*

By this means the operation No. 1 will be extended to the following: thus,

$0+3=3$, and $3\times\cdot4=1\cdot2$
$1\cdot2+4=5\cdot2$ and $5\cdot2\times\cdot4=2\cdot08$
$2\cdot08+5=7\cdot08$ and $7\cdot08\times\cdot4=2\cdot832$
&c.

3	4	5	6	7
0	12	208	2832	35328
3	52	708	8832	105328
				divisor

The law for placing the numbers being evident, I shall drop the use of the decimal point, and proceed with the next part of the rule.

4. *Multiply the divisor by the new figure of the root; subtract the product from the absolute number, and annex as many of the next remaining figures to the remainder as there are units in the exponent of the root.*

By this means we shall have the beginning of the operation No. 2.

Here the divisor, 105328, multiplied by 4, gives 4·21312 for the first subtrahend.

No. 2
5·0400084375
4·21312
8268884375

In the remaining part of the operation the law of placing the figures is evident, without the decimal point.

5. *Multiply any given term, in any class, by the last figure of the root; add the product to the term above that which is to occupy the opposite or corresponding place in the next right-hand class, and the sum is the opposite term itself in that place; observing, that if the number above be the sum of a class, to add the product to the term next above that sum.*

Note. — *The number of terms found in this manner, in the second and third classes, must be two less than the*

number which indicates the exponent of the root to be extracted; and that any following class, after the third, must have one term less than the leading class.

Thus the operation No. 1, extended by this rule, will be

$4\times0+12=12$
$4\times12+208=256$
$4\times256+2832=3856$
&c.
Also,
$4\times12+256=304$
$4\times304+3856=5072$
&c.

3	4	5	6	7
0	12	208	2832	35328
3	52	708	8832	105328
0	12	256	3856	50752
0	12	304	5072	
0	12	352		

6. *Add the two terms of the last class together, and divide the remainder, with a cipher annexed, by the sum; then the first figure of the quotient is the new figure of the root.*

Thus, in the present example,

$$\frac{826880}{156080}=5+;$$

therefore 5 is the new figure.

Annex the new figure to the last; multiply the number thus formed by the coefficient of the first term of the proposed equation, and place the product under the preceding terms of the second class, one figure to the right.

By this means the operation No. 1 will now be extended to the following:

Here the last figure of the root is ·4, and the new figure ·05; and their sum is ·45. Now $\cdot45\times3=\cdot135$, which, by reason of the decimals, place one figure to the right of those above.

3	4	5	6	7
0	12	208	2832	35328
3	52	708	8832	105328
0	12	256	3856	50752
0	12	304	5072	
0	12	352		
0	135			

8. *Add the terms of the second class together; multiply the sum by the new figure of the root; place the product under the third class, two figures to the right, and so on; and the sum of the last class will be the next divisor.*

By this means the operation No. 1 will now be extended to the following:

3	4	5	6	7
0	12	208	2832	35328
3	52	708	8832	105328
0	12	256	3856	50752
0	12	304	5072	
0	12	352		
0	135	5075	835375	92976875
3	1015	167075	18595375	1653776875

Thus the sum of the second class is 1015, which, multiplied by the new figure 5 of the root, the product is 5075; and so on for the other classes.

9. *Multiply the divisor by the new figure of the root, and the product is the subtrahend; subtract the subtrahend from the resolvend, and this completes the second step of the work.*

By this means we have the following extension of No. 2:

Thus 1653776875 is the divisor, which, being multiplied by 5, the new figure, gives 8268884375 for the subtrahend; and here, as there is no remainder, the work is finished.

5·0400084375
4·21312
8268884375
8268884375

10. *If there had been any remainder, we must have proceeded in the same manner with the third step, &c.*

In the following examples, the first class, for the sake of brevity, is placed above the root; and the two branches of the operation are put side by side of each other.

The sum of the first two numbers in the last class will be useful, in the first step or two of the operation, as a

trial divisor, in order to ascertain the new figure; but when the process has advanced, so that the real divisor has acquired one or more constant figures, it will not be necessary to use the trial divisor, and still less so as the work proceeds.

N.B.—If any class consist of a series of ciphers, the next right-hand class will have the opposite numbers each the same as the last number of the corresponding class in the last step: for this reason the first class of every step is omitted in the following examples.

Example I.

Find the root of the quadratic equation $3x^2+5x=11$.

Here the involved part of the operation only consists of one class in each step, arranged in the second column. The first column is omitted in this and the following examples, as it consists in a repetition of the coefficient of the first term of the proposed equation.

5	
3	(3)
8	11(1.2549
36	8
116	300
75	232
1235	6800
162	6175
12512	62500
147	50048
125261	1245200
	1127403
	117797

Example II.

Find the root of the cubic equation $3x^3+5x^2+7x=21$.

In this example, and in the next, each step of the operation consists of two classes.

		(3)
5	7	21(1·199
3	8	15
8	15	·6000
3	11	2743
33	143	3257000
143	2743	2722977
3	146	534023000
57	13653	286078797
1517	302553	247944203
27	13896	
297	141633	
15737	31786533	

EXAMPLE III.

Extract the root of $x^3+3x^2+5x=1881718027578170\cdot433$.

Here, by pointing the absolute number into triplets, from the place of units, there are five sections, besides the unit left on the left-hand; and, since the cube root of 1 is 1, therefore the first denomination of the root is likely to be 100000; and by proceeding with the operation this will be found to succeed.

		(1)
3	5	1881718027578170·433(123456·7
1.....	100003.....	10000300005
100003	10000300005	88168802707
1	200003	72801320010
12	640006	153674826978
320003	36400660005	132869187015
2	680006	208056399631
23	1089009	182141996820
363003	44289729005	259144028117
3	1098009	228509665275
34	1477612	306343628420
369403	45535499205	274336022154
4	1479212	32007606266433
45	1851265	32007606266433
370253	45701933055	
5	1851515	
56	2222154	
370359	45722670359	
6	2222190	
67	25926019	
3703717	4572515180919	

Whence $x=123456\cdot7$.

N.B.—If the remainder were to continue in every step, the form of each succeeding step of the process would be the same as the last step.

In this example, the figures of the root being entirely integral, may occasion some little difficulty at first in placing the terms of each respective class: a little reflection, by supposing the proper number of ciphers annexed, will, however, soon remove the difficulty, and show the law of placing them.

Example V.

Extract the root of the equation $x^3+3x^2+4x=-5$.

3	4
—2	—2
1	2
—2	2
—22	64
—32	464
—2	68
—21	361
—361	53561
—1	362
—13	10899
—3633	5403199

(1)
$-5\ (\bar{2}\cdot\bar{2}\bar{1}\bar{3}$
—4
—1000
— 928
—72000
—53561
—18439000
—16209597
—2229403

Whence $x=-2\cdot213$.

Example VI.

Extract the root of the equation $x^3-5x^2+7x=1$.

In this example, in order to prevent subtractions, the arithmetical equivalent to —5 is used; thus, $-5=\dot{-}10+5=\bar{1}5$; by this means the work is not interrupted by subtractions.

$\bar{1}5\cdot$	7
$\cdot1\cdot$	$\bar{1}\cdot51$
$\bar{1}5\cdot1$	$6\cdot51$
1	$\bar{1}52$
16	$\bar{1}7216$
$\bar{1}536$	57516
6	$\bar{1}7252$
607	$\bar{1}688649$
$\bar{1}54807$	547363649

(1)
$1\cdot000(\cdot1607$
651
349000
345096
3904000000
3831545543
72454457

Example VII.

Extract the root of

$$x^3-7035x^2+15262754x=10000730880.$$

$\bar{1}2965$	15262754
3	$\bar{1}87895$
$\bar{1}5965$	3157754
3	$\bar{1}96895$
34	9460
2365	998754
4	11060
45	16075
3215	2265504
5	16325
56	19926
3321	2448680

10000730880(3456
9473262
5274688
3995016
12796728
11327520
14692080
14692080

Whence

$x=3456.$

The arithmetical equivalent is here used for the same reason as in the former example.

Example VIII.

Required the square root of 4654389657.

In this and the following examples the coefficients of all the powers, except that of the highest, which is unity, are each zero; therefore, instead of these coefficients, ciphers may either be introduced or omitted altogether.

6	. . . . . (1)
6	4654389657(68223
68	36
128	1054
82	1024
1362	3038
22	2724
13642	·31496
23	27284
136443	·421257
	409329
	11928

The general method coincides here with the square root. Hence it applies equally to finding the root of numbers.

Example IX.

Required the cube root of the number 311897910.

0	0
6	36
6	36
6	72
67	1309
187	12109
7	1358
78	16144
2018	1362844
8	16208
81	20341
20341	137925541

311897·910(67·81
216
95897
84763
11134910
10902752
232158000
137925541
94232459

Example X.

Required the fourth root of 5202888229840401.

8	64	512
8	64	512
8	128	1536
8	192	
84	1296	158784
324	39696	2206784
4	1312	164032
4	1328	
49	30321	38375289
3369	4263921	2409191289
9	30402	38648907
9	30483	
98	101889	1297747467
33963	432582489	2449137943467

5202888229840401(8493
4096
11068882
8827136
2241746298
2168272160
73474138304
73474138304
0

Example XI.

Extract the fifth root of the number 441101415279249.

8	64	512	4096
8	64	512	4096
8	128	1536	16384
8	192	3072	
8	256		
84	1616	262464	21529856
404	65616	5382464	226529856
4	1632	268992	22605824
4	1648	275584	
4	1664		
49	37881	63841929	53017964361
4209	7093881	5990884929	2343274764361

441101415279249(849
32768
1134214152
905319424
22889472879249
22889472879249

EXAMPLE XII.

Extract the root of $x^4+5x^3+4x^2+3x=105$ *to four places of figures, and extend the root by the contracted rule.*

5	4	3	105(2·217
2	14	36	78
7	18	39	270000
2	18	72	246256
2	22		237440000
22	264	12128	136453781
132	6064	123128	1009862190000
2	268	12664	963079632921
2	272		46782557079(3
21	1381	661781	41420810748
1381	661781	136453781	5361746331(3
1	1382	663163	4142741941
1	1383		1219004390(8
17	96929	465860703	1104751479
13847	665515 29	137582804703	114252911(8
7	969 78	466539549	110475617
7	970 27		3777294(2
7	41|60	2002490|8	2761891
13|868	667496|94	13806936916|0	138094)1015403(7
	41|60	2002615|6	966661
	41|60		48742(3
	|03 9	20027|5	41428
|013	6675|84	1380913980|6	7314(5
	|03	20027|6	6904
	|03	534|0	410(2
	66|759	1380939334|9	276
		534|0	134(9
		5|34	124
	|66	13809452|2	10(7
		5|3	9
		138094|5|7	1

Whence $x=2·21733882735297+$

In order to understand the contracted part of this operation, see the observations and Rule in p. 30.

EXAMPLES FOR PRACTICE,

WITH THEIR ANSWERS,

Which were all found by the foregoing new methods of extracting the roots of equations and numbers, most of the operations by which the answers were obtained being performed by the Author himself, in the year 1818, and in Jan. 1819, and afterwards confirmed by his pupils.

Extract the Roots of the following Equations and of the following Numbers.

Answers.

1. $x^3+12x^2+1728x=45$ .... $x=\cdot026036948658$
2. $x^3-171\cdot91x^2+7905\cdot6x=72694.5678$.... $x=12\cdot20420601$
3. $x^3+30x=420$......................... $x=6\cdot170103$
4. $x^3=484476471864$......................... $x=7854$
5. $x^3-22x=24$.................... $x=5\cdot1622776601665$
6. $x^3-9x=12$............................ $x=3\cdot52233$
7. $x^3-2x=5$.......................... $x=2\cdot09455148$
8. $x^3+x^2+x=13099751099$.................... $x=2357$
9. $5x^2+3x=100$.......................... $x=4\cdot182186$
10. $7x^2+3x=7854326416$.................... $x=33496$
11. $47653x^2+3654x=237$.................. $x=\cdot006378$
12. $144x^3-973x=319$......................... $x=2\cdot75$
13. $x^3+7x^2+3x=57$.................... $x=2\cdot31759722$
14. $9x^3+7x^2+5x=547$.................. $x=3\cdot64481861$
15. $55x^2+176x=59$....................... $x=326\cdot9501$
16. $x^3+5x^2+7x=54387692$................. $x=377\cdot21$
17. $x^3+9x=500$.......................... $x=7\cdot0525584$
18. $9x^3+5x^2+x=155$........................ $x=2\cdot39658$
19. $x^3-12x=15$............................ $x=\begin{cases} 3\cdot971963, \text{ or} \\ -1\cdot577032, \text{ or} \\ -2\cdot394930 \end{cases}$
20. $x^3-6x=2$.................................... $x=\begin{cases} 2\cdot601676 \\ -2\cdot261806 \\ -\ \cdot339870 \end{cases}$

Answers.

21. $x^3+9x=3$ .... $x=2\cdot180849$

22. $x^3-2x=5$ .... $x=2\cdot0945514$

23. $x^3-23x=-16$ .... $x=-5\cdot472136$

24. $x^3-27x=36$ .... $x=\begin{cases}5\cdot765722\\-4\cdot320684\\-1\cdot445038\end{cases}$

25. $x^4-19x^3+132x^2-302x+200=0$ .... $x=\begin{cases}1\cdot02804\\4\cdot00000\\6\ 57653\\7\cdot39543\end{cases}$

26. $x^4-27x^3+162x^2+356x-1200=0$ ..... $x=\begin{cases}2\cdot05608\\-3\cdot00000\\\cdot13\cdot15306\\14\cdot79086\end{cases}$

27. $x^4-12x^2+12x-3=0$ .... $x=\begin{cases}\cdot606018\\-3\cdot907378\\2\cdot858083\\\cdot443277\end{cases}$

28. $x^4-38x^3+210x^2+538x+289=0$ ...... $x=30\cdot53565375$

29. $x^5+6x^4-10x^3-112x^2-207x-110=0$ .... $x=4\cdot46410161$

30. $x^5+2x^4+3x^3+4x^2+5x=54321$ .... $x=8\cdot414455$

31. $*7x^4-11x^3+6x^2+5x=215$ ...... $x=2\cdot70648049385791$

32. $*3x^5+11x^3+17x=189$ .... $x=1\cdot920663752601$

33. $**7x^5+6x^4+5x^3+4x^2+3x=11$

Ans. $x=.77076881962265852237929650 5$

34. $*4x^6+7x^5+9x^4+6x^3+5x^2+3x=792$

Ans. $x=2\ 052042176879505365214043401281201973460275599545541724214$

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* The three examples that have the single star prefixed were performed by Mr. Joseph Seers, a pupil of mine, a very industrious young man, who has made considerable progress in the mathematics, particularly algebra.

** The example that has the two stars prefixed was performed by Mr. James Wilson, another pupil of mine, and a very excellent arithmetician. The very great number of figures found in the root, and that are here exhibited, shows the facility of the method now proposed for extracting the roots of numerical equations.

END OF EVOLUTION.

POSTSCRIPT

To the

Theory

OF

INVOLUTION AND EVOLUTION.

MR. HOLDRED having at last published his long projected work on the resolution of equations; and having adopted various important improvements suggested to him by me, during the time I was engaged in the inspection of his original manuscript; it becomes necessary for me to offer a few observations upon his work. And this the more especially, as, far from acknowledging what he has introduced in consequence of my communications, he has laboured to undervalue my suggestions, by treating them as trifling and useless.

In justice to myself, I shall therefore select a few passages from his work, and subjoin such comments as will place this matter in a true light.

In the preface, dated June 1st, 1820, Mr. Holdred informs us of the motives that led to the invention, in the following terms:—"The first discovery of this method was "made when I was about twenty years of age, now forty "years ago. Regretting that a method for extracting the "cube root should be so troublesome, I at length thought "of forming a canon for finding the square of the root as "it became increased, with the addition of every new figure, "as they became known, considering it as a binomial root; "the first member being the known part of the root, and "the other member the newly-discovered figure. This "canon being formed, and multiplied by 3, it was easy to "perceive that the preceding divisor being singly taken, the "second member being multiplied by the newly-discovered "figure, and then doubled, with the treble square of the "said newly-discovered figure, made up the canon. I then

" considered, that, in extracting the square root, the double " of the known part of the root is taken for the divisor, to " which, if the double of the newly-discovered figure be " added, the sum will be the double of the increased root " for a new divisor. By this means I saw that the method " would apply to mixed equations, and that the sums of " the different powers of the increased root are had out of " the sums of the respective powers of the preceding part " of the root.

" I then applied the same principle to the biquadrate " root with equal success; and perceiving that the coeffi- " cients were the beginning of the figurate numbers, I " concluded that it must be universal, which (by means of " Sir Isaac Newton's Binomial Theorem) I soon found to " be the case."

The account which Mr. Holdred here gives us is rational and ingenious, though not a process of direct investigation. It shews the steps by which he arrived at the generalization of the principle.

He then proceeds as follows: " There may be various " ways of writing down the numbers in an operation: my " chief aim has been to be clearly understood; for which " purpose the algebraic symbols are used throughout the " work, with the exception of the Supplement."

In the Introduction to my Algebra, page 6, I stated the improvements on Mr. Holdred's method, which I had then made to consist,—first, in freeing the process entirely of algebraic characters and symbols, and thereby changing the form of calculation into a very concise operation, purely arithmetical, and uniform in all its steps; and secondly, in explaining the law by which the similar parts might be performed without the continuation of decimal fractions to the end of the operation.

In my Treatise on Involution and Evolution I re-stated these improvements in words to the same effect.

From what I there said it must not be supposed that, by freeing the process from algebraic characters and symbols, I meant to exclude those which were necessary to the general demonstration. What I laid claim to was the idea of giving to the incipient step of the method the very same form as was employed in all the succeeding steps, and the

mode of disembarrassing the operation from a multitude of minor algebraic equations, expressing the value of certain coefficients; which improvement I effected by deriving one part from another, according to arithmetical rules only. This will be more fully explained in the sequel.

Mr. H. proceeds again. "In June, 1818, I submitted "to Mr. P. Nicholson a manuscript of this tract, which "met with his entire approbation; and he gave me the "following recommendation in writing, which I inserted in "the Prospectus:—

Mr. Theophilus Holdred has submitted to my opinion a tract on the resolution of equations of all degrees. I have perused it with care, and have found it to be a most ingenious method; for, notwithstanding the numerous attempts and industry of the most eminent mathematicians, from the time of Cardan to the present, to discover an easy, direct, and correct process of extracting roots, this method eluded their research, and has been reserved to raise the fame of an individual hitherto unknown in the mathematical world.

PETER NICHOLSON.

No. 12, London Street, Fitzroy Square,
October 7th, 1818.

The essence of the principle which Mr. Holdred had discovered certaiuly met with my entire approbation. But neither his demonstration, nor the manner in which his practical operations were performed, would, in my opinion, have been intelligible to any class of readers.

In my treatise on Involution, &c. page iv, I stated that, after I had pointed out many defects and obscurities in his manuscript, he agreed to write the whole anew under my inspection, and to adopt such farther improvements as might occur during the period of re-writing it. On this subject Mr. H. says, "Mr. Nicholson recommended a different "notation, with some other alterations of little importance; "and *though I did not perceive any advantage in it, I* "*yielded to his advice, and the manuscript was written over* "*again:* this, on account of my profession, was not com- "pleted before December."

I certainly did recommend a different notation, and some other alterations of little consideration; but these I thought too trifling to be worth enumerating among the improve-

ments which I laid claim to. He has wholly forgotten to mention all those points that were of real importance, and which I freely suggested to him as they occurred to me. Most assuredly, if his treatise had been published from the original manuscript, and without my communications and improvements, it would have been utterly unintelligible.

Among the improvements of minor importance of which Mr. H. has availed himself, I shall just particularize the introduction of *arithmetical equivalents*, where any of the coefficients of a proposed equation are negative, and the notation of figurate numbers and factorial factors, or factors succeeding each other in arithmetical progression. These subjects form Sections I and V of Mr. H.'s tract.

This last hint has of itself been of considerable utility to Mr. Holdred's work. The general demonstration given in Section VI of his publication is, even in its present state, any thing but *simple* and *easy of apprehension*. But under the form in which it appeared in his first manuscript, the figurates and factorials, instead of possessing their present elegant and highly condensed notation, were written at length, and presented an aggregation of algebraic symbols capable of intimidating and revolting the most determined admirer of analytical researches.

In short, if the alterations and improvements which I suggested to him were of the little importance that he pretends, is it to be imagined that he would have consented to write the work anew, and by that means have so long delayed the publication of it? And this, too, in the face of a declaration that he had matured this favourite child of his youth by forty years' meditation and experience.

In the Introduction to Involution, &c. already alluded to, I state in page vi, that Mr. Theophilus Holdred, a gentleman but little known to the mathematical world, some time since submitted to my inspection and opinion an original tract, containing a method of finding the roots of equations of all degrees in numbers; but that from the obscurity, want of connection, and antiquated manner in which the subject was treated, I was able to form but a very imperfect idea of the principles upon which his method was founded. Anxious, however, to accomplish what had been deemed by the first mathematicians a matter of the

utmost importance, I resolved not to lose sight of so desirable an object. Without attending to the manner in which Mr. Holdred had considered the subject, but keeping steadily in view Newton's principle of approximation, I soon conceived, that, to extract the root of an equation in numbers, it was requisite to find a series of transformed equations of such a nature, that, in every two consecutive equations, the root of the former should be diminished by a single digit or denomination, which should have the greatest value possible, not exceeding the root; and also that the process of transformation should be performed by arithmetical rules instead of by the binomial theorem, as had hitherto been practised.

Having discovered the manner in which these desirable ends were to be obtained, together with the demonstration of the theory, I communicated the result to Mr. Holdred, who admitted the simplicity of the principle, allowing it to be more concise, and easier of comprehension, than his own; and that it led immediately to the rule, without circuitous steps in the demonstration.

Mr. H.'s account is as follows:—

"Mr. Nicholson discovered another way of demonstrat-
"ing my rule, which he requested me to add to my treatise
"by way of supplement; but finding that he was com-
"municating my method to every mathematician he knew,
"I became dissatisfied with his conduct; and discovering
"the improvement which I have inserted as a Supplement,
"I considered it far better than that intended by Mr. Nichol-
"son; I therefore resolved to prefer my own, which I have
"never communicated to him."

The general demonstration which I discovered is much superior to the one he has given, as being not only more obvious, but also contained in much less compass. It occupies only three octavo pages (*see* my Essay on Involution, &c.), while his demonstration requires eight quarto pages.

The improved method Mr. Holdred professes to have invented, and which forms the subject of his Supplement, he has made to depend wholly upon the clumsy and imperfect demonstration given in the first part of his treatise; I must confess, therefore, that I do not see how his Sup-

plement can well be considered as a substitute for the one proposed by me.

He speaks of this improvement in terms of the highest exultation, and, for aught I know to the contrary, it may be his own; but in both the discovery and publication of it, he has been anticipated by Mr. Horner, of Bath; and, in point of publication, he has also been anticipated by me. It is, therefore, contrary to custom and experience to suppose that the world will easily be induced to give him the credit of this last improvement. Moreover, he has left it in so unfinished a state, that few will be inclined to study the method under the form delivered in his Supplement. Let any unbiassed and disinterested person compare the result in page 57 of my Essay with that in page 51 of his Tract, and I imagine he will not hesitate to which the preference should be given.

The charge of my having shewn his method to every mathematician I knew, is not altogether true. The world has been so often deceived by persons pretending to have discovered general methods of extracting the roots of equations, that, without some evidence which could be relied on, few would be disposed to listen to any such claims; and particularly when proceeding from an obscure individual. My sole desire in taking up the subject was to serve him and promote the general interests of the science. I therefore explained the method to such of my acquaintance as were likely to become subscribers, and were possessed of sufficient weight and influence to recommend his work to others. I most certainly had a right to do this, since I had a great share in the improving of the method, both the demonstration and practical rules being entirely different from his. To understand this rightly, the Reader must observe that the operations, as he has published them, are not those of his original manuscript, but those which I communicated to him.

Though the principle first originated with him, I can positively assert, that what he called his demonstration was so undigested, obscure, and antiquated, that I did not receive a single idea from it. I investigated the principle, and digested the rules, as if nothing of the kind had been

done. Whoever will give themselves the trouble of comparing the *Essay on Involution and Evolution*, written by me since our intercourse has ceased, with the tract Mr. Holdred has now published, will find this latter publication does not contain a single idea or improvement which has not been anticipated in my work; but, on the contrary, they will perceive many improvements in my treatise that are not to be found in his.

Mr. Holdred proceeds—"After the flattering testimony "which Mr. Nicholson gave of the advantages of my "method, is it not surprising that he should have pub- "lished a treatise on Algebra, obviously for the purpose of "introducing that method? My subscribers well knew how "long my manuscript had been completed, as well as the "unavoidable causes which procrastinated its publication: "let them read Mr. Nicholson's recommendation, and then "draw their own conclusions."

With respect to my treatise on Algebra, I can produce the most flattering evidences of the approbation with which several parts of it have been honoured by mathematicians of the first eminence; and this in articles which do not relate to the extraction of roots. That the Algebra was not expressly published for the purpose of introducing this method of resolving equations, I can easily make appear. My Architectural Dictionary contains numerous articles on geometry, mechanics, and other branches of mathematics; and to enable the student to understand these without being under the necessity of recurring to other works, I considered it requisite to introduce an article on algebra: this was printing at the time the intercourse between Mr. Holdred and me was broken off; and as another work on the subject of equations was at that time much talked of, I did not hesitate to insert a sketch of the new principle of extraction, according to my own improvements, and in a form entirely different from that in Mr. H.'s first manuscript. As I had bestowed considerable pains in drawing up the article on Algebra, and had suggested many improvements not elsewhere to be met with, I thought that the mere introduction of the article into my Architectural Dictionary was confining it within narrower limits than it deserved; I therefore persuaded the publisher, Mr. Barfield, to make a dis-

tinct work of the Algebra; and accordingly it was printed from the same composition as in the Dictionary, without the type being distributed.

I published the method with my own improvements, stating that Mr. Holdred was the inventor of the principle. If this had not been done, the credit of the discovery must have been given to Mr. Horner of Bath, who wrote a tract upon the same subject, which was read to the Royal Society on the 1st of July, 1819, the very same day on which my Algebra was distributed among the subscribers to it.

Mr. Holdred says, "Let them read Mr. Nicholson's "recommendation, and then draw their own conclusions." After considering the extracts which I have given from Mr. Holdred's Preface, and my comments upon them, I too am content that my readers shall draw their own conclusions.

It is something singular that Mr. Holdred should have made no mention of what Mr. Horner has done; though I informed his friend Mr. Gibson, Watch-maker, at Hampstead, of the circumstance of its publication in the Philosophical Transactions, as soon as I knew of it; particularly as Mr. Gibson is greatly interested in Mr. Holdred's behalf, and was to have written the Preface to his method. In justice to Mr. Horner this should have been done, since the improvement which forms the subject of Mr. Holdred's Supplement first occurred to Mr. Horner.

It is also remarkable that Mr. Holdred has not mentioned my Essay on Involution and Evolution, which contains the improvement he was so anxious to conceal, and that in a form much better adapted to arithmetical calculation than the one introduced into his Supplement. It may be possible that he has never seen my Essay, but from the two following reasons it is *very* improbable. First, When speaking of me in his Preface, in order to undervalue what I had done, several of his statements have no reference to any part of the Algebra, but relate to my Essay on Involution and Evolution, as may be seen by comparing what I have extracted from my Essay and from his Preface. And secondly, Mr. Gibson, his zealous and intimate friend, is known to have purchased at the shop of Messrs. Davis and Dickson, in Saint Martin's-le-Grand, a copy of my Essay, several weeks before the publication of Mr. Holdred's work, which

was about the 15th of June, 1820. The work was certainly not purchased for Mr. Gibson's own use; as, though he is an ingenious mechanic, and possesses a thorough knowledge of orthographical and perspective projections, he has little or no acquaintance with even the elements of algebra.

Mr. Holdred concludes as follows: "This last method "(that of the Supplement) I conceive to be the perfection "of the whole, as there is no occasion for the arithmetical "equivalents of negative numbers; and it is only necessary "to write down the proper signs. It is admirable for its "simplicity, as any child who can multiply and divide, "may perform it with ease, the number of multiplications "being greatly reduced."

As to the subject of his Supplement, which has been laboured together with very little science, and which he says is the "perfection of the whole," it is most probably not his own; not only for a reason which I shall hereafter state, but from his knowledge of its being already published, both by Mr. Horner and myself; and more particularly as my Essay was perfectly adapted to his reading, and was published a sufficient time for him to have availed himself of its contents. His object in producing this method has been as much to dispense with the arithmetical equivalents as with the figurate factors; and thus to make himself as independent of my suggestions as possible. But it is singular that he has retained these mixed numbers throughout his whole work, except the last Example of his Supplement; and, at all events, allowing that he had neither availed himself of my communications nor of the two works that were published before his on the same subject, it could not be wondered that, after forty years' close meditation on the subject, he should be able to effect this last object in a certain manner.

The circumstance of Mr. Holdred's being able to lay aside the use of the arithmetical equivalents, arises from his constantly summing in pairs the numbers forming the operation. This method, far from being an advantage, actually adds as many additional lines to each complete step as there are units in the index of the exponent of the equation.

MR. NICHOLSON'S METHOD COMPARED WITH MR. HOLDRED'S.

Find the value of x in $x^4+5x^3+7x^2+3x=91672020000$, being Example 6th, page 21 of Mr. Holdred's new method of solving equations.

Solution by Mr. Nicholson's Figurate Method, published July 1st, 1819, in his Algebra.

			$3=D$	
(500)		7	$35..=Ca$	
	5	25..	$125....=Ba^2$	
1	5..	25....	$125.....=Aa^3$	
			$126253503=\delta$	91672020000(549
			$503757003=D_2$	$631267515=\delta a$
(40)		1507507	$6030028=C_2b$	2854526850
	2005	8020	$32080=B_2b^2$	$2269317132=\delta_2b$
1	4	16·	$64=A_2b^3$	5852097180
			$564329283=\delta_2$	$5852097180=\delta_3c$
			$654237563=D_3$	
(9)		1757707	$15819363=C_3c$	
	2165	19485	$175365=B_3c^2$	
1	9	81	$729=A_3c^3$	Whence $x=549$
			$650233024=$	

Here, in order to render the law of placing the numbers clear, points are used in the first step instead of ciphers. The right-hand figure of any number which stands under another in any step is as many figures to the left of the number above, as there are ciphers after the multiplying figure. For instance, in the second step the multiplier is 40, which has one cipher; therefore the right-hand figure of any following number stands under the second figure from the right of the number above.

Any class of numbers on the right is derived from that on the left, except the coefficient which stands on the top; thus, by multiplying 1, which is the first class of the first step, by 500, we get the second class, as in the margin, by placing the coefficient of the second term above it.

5
5..

Again, by multiplying the numbers of the second class above by 500, we get the third class, as in the margin, by placing the coefficient of the third term above the numbers thus multiplied; and so on.

07
·25.
25

But the greatest difficulty is to find the coefficients of the new equation. It will be sufficient to shew this in the last class of the first step.

For this purpose let it be repeated as in the margin. Begin at the first adding column, and write down 3, place a cipher for the sum of the second; multiply 5 by the figurate 2, set down a cipher for the sum of the third, and carry 1; multiplying 3 by the figurate 2 gives 6, and 1 carried is 7, which write down in the fourth place; then multiplying 5 by the figurate 3 gives 15, set down 5 and carry 1; then 3 times 2 is 6 and 1 carried is 7, which set down. Again, say 4 times 5 is 20, and 3 times 1 is 3, then $20+3=23$, set down 3 and carry 2. Again, say 4 times 2 is 8, and 2 carried is 10, set down a cipher and carry 1. Lastly, 4 times 1 is 4, and 1 carried is 5; set down 5 in the last place, and write the sum 503757003 under the divisor; and so on for the other classes.

$$\begin{array}{r} \cdot03\times1 \\ \cdot35..\times2 \\ 125....\times3 \\ 125......\times4 \\ \hline 503757003 \end{array}$$

The same Example as that on the opposite side, from p. 21 of Mr. Holdred's new method of solving equations, as improved by my suggestions to him when he wrote his second manuscript under my inspection, published about the 15th of June, 1820, the Preface to which is dated June 1, 1820 :—

Let the equation $x^4+5x^3+7x^2+3x=91672020000$ be proposed.

Make $r=500$.

$r^4=62500000000$	$4r^3=500000000$	$6r^2=1500000$	$4r=2000$
$hr^3=\ \ 625000000$	$3hr^2=\ \ 3750000$	$3hr=\ \ 7500$	$h=\ \ 5$
$ir^2=\ \ 1750000$	$2ir=\ \ 7000$	$i=\ \ 7$	$D=2005$
$kr=\ \ 1500$	$k=\ \ 3$	$C=1507507$	${}_2D=D+4a$
$A=63126751500$	$B=503757003$	${}_2C=C+3Da+6a^2$	
	${}_2B=B+2Ca+3Da^2+4a^3$		

$N=91672020000(500=r$
63126751500
28545268500
$B=503757003)$

$C=1507507 : \times 40 = 60300280 = Ca$

$D=2005 \quad 3Da = 240600 \qquad 3208000 = Da^2$

$Da=80200, \quad 4a \ \ 160 \quad 6a^2 = \ \ 9600 \qquad 64000 = a^3$

$a^2 = 1600 \qquad$ Divisor $567329283 : \times 40 =$

${}_2B=634237563$

$C=1757707 : \times 9 = 15819363 = {}_2Cb$

${}_2D=2165 \qquad 175365 = {}_2Db^2$

${}_2Db=19485 \qquad 729 = b^3$

$b^2 = \ \ 81 \qquad$ Divisor $650233020 : \times 9 =$

This operation contains 100 figures more than that on the opposite side, where no multiplier is used higher than 9; but in the above, higher multipliers are indispensable. In the opposite process, every succeeding part is derived from that which precedes it; viz. the right-hand numbers are derived from those on the left; but this is not generally the case in the process above. As to the numbers in the last or right-hand class, the arrangement is the same in both; as also the operation of finding the divisors and trial divisors. The first step of Mr. Holdred's work, as above, is very complex, and far from being uniform with the remaining two steps; but in my operation the first step is exactly similar to all the succeeding steps.

The following is one of Mr. Holdred's Examples, taken from his first manuscript, as put into my hands:—

In the equation $x^5-5x^3+5x=N$. Make $N=1=$ the chord of 60°; then the value of g being found to be equal to 2,

$$\begin{aligned}
g&=\cdot 2\\
g^2&=\cdot 04\\
g^3&=\cdot 008\\
g^4&=\cdot 0016\\
g^5&=\cdot 00032\\[1ex]
e&=9\\
De&=414\\
Ee&=\cdot 09\\
Ee^2&=\cdot 81
\end{aligned}$$

$$\begin{aligned}
g^5&=+0\cdot 00032\\
5g&=+1\cdot 00000\\
g^5+5g&=+\overline{1\cdot 00032}\\
5g^3&=-0\cdot 04000\\
g^5-5g^3+5g&=+\overline{0\cdot 96032}=A\\[1ex]
5g^4&=+\overline{0\cdot 00800}\\
5&=+5\cdot\\
5g^4+5&=+\overline{5\cdot 00800}\\
-15g^2&=-0\cdot 60000\\
5g^4-15g^2+5&=+\overline{4\cdot 40800}=B
\end{aligned}$$

$$\begin{aligned}
10g^3&=+0\cdot 080\\
15g&=-3\cdot 000\\
10g^3-15g&=-\overline{2\cdot 920}=C\\
10g^2&=+\overline{0\cdot 4}\\
&\;\;-5\cdot 0\\
&\;\;\overline{-4\cdot 6}=D\\
5g&=+\overline{1\cdot 0}=E
\end{aligned}$$

$$\begin{aligned}
B&=44080000\\
4Ee^3&=\quad 29\\
B+4Ee^3&=\overline{44080029}\\
2Ce&=-525600\\
3De^2&=-\;11178\\
2Ce+3De^2&=\overline{-536778}\\
{}_1B&=\overline{43543251}\\[1ex]
C&=-\;29200\\
3De&=-\;1242\\
3De&=\overline{-\;30442}\\
6Ce^2&=+\quad 4\\
{}_1C&\;\overline{-\;30438}
\end{aligned}$$

$$\begin{array}{rr|l}
 & N=1\cdot 0000000000 & (0\cdot 2090569265= \\
 & 96032 & \text{the chord of 12 degrees} \\
\text{Resolvend} & =+\;396800000 & \\
 & C=-\quad 29200 & e=9 \\
 & D=-\quad 46 & \\
 & E=+\quad & 01 \\
 & B=\;+44080000 & \\
 & Ee^3=\quad 7 & 3 \\
 & Ce=-\;262800 & \\
 & De^2=-\;3726 & \\
 & Ce+De^2=-\;266526 & \\
\text{True divisor} & =\;43813481 & 3 \\
\text{Subtrahend} & =\;394321332 & \\
\text{Resolvend} & =\;2478668 & \\
 & {}_2C=-\quad 3 & 04 \\
 & {}_2B=+\;435432 & 5 \quad e=5 \\
 & -15 & 2 \\
\text{True divisor} & =\;435417 & \\
\text{Subtrahend} & =\;2177086 & \\
\text{Resolvend} & =\;301582 & \\
 & - & 03 \\
 & 43540 & 2 \quad e=6 \\
 & - & 2 \\
\text{True divisor} & =\;43540 & 0 \\
 & 261240 & \\
 & 4354)40342 & (9265 \\
 & 39186 & \\
 & 1156 & \\
 & 871 & \\
 & 285 & \\
 & 261 & \\
 & 24 & \\
 & 22 & \\
 & 2 &
\end{array}$$

The same Example as that on the opposite page, now published by Mr. Holdred, about the 15th June, 1820, with my improvements, being his 11th Example. *See* **pages 40 and 41.**

In the equation $x^5-5x^3+5x^3+5x=N$; make $N=1=$ the chord of 60 degrees; and by writing the arithmetical equivalent for the negative coefficient, the equation becomes $x^5+15x^3+5x=1$; make $r=0{,}2$

$r^5=0{,}00032$	$5r^4=0{,}008$	$10r^3=\ 0{,}08$	$10r^2=\ 0{,}4$	$5r=1$
$\bar{1}5r^3=0{,}\bar{1}6000$	$\bar{1}85r^2=1{,}400$	$\bar{1}85r=\bar{1}7{,}00$	$\bar{1}95=\bar{1}5{,}0$	$E=1$
$5r=1{,}00000$	$5=5{,}000$	$C=17{,}08$	$D=\bar{1}5{,}4$	
$A=0{,}96032$	$B=4{,}408$			

		$N=1{,}0000000000(0{\cdot}2090569265$
	$B=4{,}408$	96032
	$C=\bar{1}7{,}08 : a=\ \bar{1}7372$	$396800000(a={\cdot}009$
$D=\bar{1}5{,}4$	$Da^2=\cdots\bar{1}6274$	
$Da=\ \bar{1}586$	$Ea^3=\cdots\cdots729$	
	$a^4=\cdots\cdots\cdots6+$	
$E=1,$	divisor ...$4{,}381348135$:	$\times a=394321392+$subtrahend
$Ea=\ {,}009$	$B=4{,}3543251\cdot\cdot$	$2478668(b={,}00005$
$Ea^2=\ {,}000081$	$\bar{1}6{,}956 : \times b=^*\ \ \bar{1}8478$	
	divisor...$4{,}3541729$:	$\times b=\ 2177086+$subtrahend
$B=4{,}408$	$_3B=4{,}35402\cdot\cdot$	$301582(c=000006$
$2Ca=\cdot\bar{1}4744$	$_2Cc=\ \ \bar{1}8$	
$3Da^2=\cdot\cdot\bar{1}88822$	$4{,}35400$:	$\times c=\ 261240+$subtrahend
$4Ea^3=\cdots\cdots2916$		$40342(9265$
$5a^4=\cdots\cdots\cdot32$		39186
$_2B=4{,}354325148$		1156
$C=\bar{1}7{,}08$		871
$3Da=\bar{1}{,}8758$		285
$6Ea^2=\cdots\cdots486$		261
$_2C=\bar{1}6{,}956'286$		24
		21
		3

I recommended the use of the notation B, C, D, &c.; $_2B$, $_2C$, $_2D$, &c.; $_3B$, $_3C$, $_3D$, &c. to represent similar values in the operation; as also the use of the letters a, b, c, &c. instead of e variable for every new figure of the root; and some other things of the like nature; but these I considered to be of little value compared to the dividing of the operation into two branches, and placing the numbers so that they might be more easily derived from each other. This separation was the natural consequence of my own demonstration, which suggested to me the best arrangement of the quantities.

The Reader will, however, perceive the difference between this last operation, which was written as I suggested to Mr. Holdred, and that in page 62, taken from his original manuscript. In this latter operation the process is almost unintelligible, from the symbols being so arranged as neither to exhibit the law of construction nor the manner in which their values can be derived from each other.

An Account of the Circumstances which have led to a NEW DEMONSTRATION, *and one of the most general and simple Formulas that has ever been exhibited; followed by* RULES *and* EXAMPLES, *which, from their Simplicity and Shortness, are without Precedent.*

IN writing the Third Essay of my Combinatorial Analysis, I had observed a singular coincidence between the formulas exhibited in pages 4 and 22 of that Essay and Figurate Numbers. I clearly saw, that if the quantities α, β, γ, &c. were each considered equal to zero, and a, b, c, &c. each equal to unity, the respective columns contained the orders of figurate numbers, and that the values exhibited in each column were those of the succeeding terms.

After I had investigated the rules for the extraction of the roots of equations, I found that, as the principle depended upon that of figurate numbers, and that as it might be considered, in one point of view, a theorem for involution, I was forcibly impressed with the idea, that, when more at leisure to mature it, I should be able to reduce the theorem for extracting the roots of equations to a similar formula as that in page 4 of the Essay now alluded to.

As I have always had the greatest pleasure in communicating what I knew, I imparted this idea, among many others, to Mr. Holdred, of which he has availed himself without acknowledgment; and most certainly the subject of his Supplement, though rudely brought together, owes its existence entirely to it.

When I published the preceding Essay on Involution and Evolution, I aimed at the reduction above alluded to; and I succeeded in deriving the values from each other in a

manner nearly similar to those in the formula intended. As to the near coincidence of the formula thus found with that given by Mr. Horner, which he has so very intricately expressed, it is merely accidental. The form in which it is exhibited in page 35, for the involution and transformation of equations, is that from which I should have drawn my rules, had I not seen his article. I altered it to that shewn in page 37 of the Essay, in order to make it coincide with his formula. By this means, though I obtained the divisor without addition, I introduced some irregularities in the notation, which made it difficult to be converted into words. But after I had published the essay which contained it, I did not feel entirely satisfied, as the values in the second line of the formula, page 35, were not formed according to the same law as the succeeding values. However, on again comparing the figurates that arose in the transformation of equations with those of the figurate orders themselves, I perceived they were alike formed; therefore, seeing no reason for this anomaly, I resolved to make another trial; and what Mr. Holdred has said concerning me in his Preface, has urged me to take this opportunity sooner than I otherwise should. I have now succeeded to my utmost wishes; and I believe that nothing more on this account remains to be done.

I have, in this Postscript, returned to the simple idea of transforming the equation, by which means the rules are easily expressed; and as I have now adopted the method of multiplying one number by a digit, and adding the product to another in one line, all the divisors appear in the last series formed by the sums, without making separate additions, as at first occurred.

From this simple idea I have arrived at the exact correspondence of the formula for involution, and the extraction of the roots of equations, with that for the transformation of binomial factors; such universality of application, uniformity of expression, simplicity of rules, and regularity of operations, are without comparison, and could hardly have been expected. Since this identity of method applies to several branches at once, it will be of the greatest advantage to the learner, as he cannot acquire any one of them without becoming acquainted with the operations of the others.

The idea of summing the values of the quantities in pairs occurred in my former demonstration, as well as in the operations; but I abandoned it on account of the multitude of figures which it introduced.

I shall now conclude by observing, that whatever advantages my practical operations possess above Mr. Holdred's, arises wholly from the nature of the demonstrations.

Mr. Holdred has neither been able to reduce his principles to a formula, nor to express his rules in words. His clumsy operations are the natural results of his methods of considering the subject, as my operations are of the demonstration and formula which I have invented.

Before I proceed farther, it may not be amiss to say a word or two on the uses and advantages of Notation.

The principle of raising a binomial quantity to any given power, without absolute multiplication, was known to algebraists before the time of Sir Isaac Newton; but the mode of expressing the rule for this purpose by symbols was first discovered by him, and has been of the utmost service to mathematicians in their investigations.

It is by superior methods of notation that foreigners have, within the last fifty years, been enabled to carry their analytical researches to an unprecedented and almost unlimited extent.

Indeed no one can say what a proper application of this most powerful instrument may not effect; it so frequently is found to group into one general formula a multitude of particular rules and cases.

The relations and similitudes which we so often discover to obtain between the expressions for different formulas are so surprising, that we are irresistibly driven to conclude them all to be a particular case of one grand and most comprehensive theorem.

This universal theorem may probably one day be discovered by the due substitution of such appropriate symbols for the expression of whole progressions of quantities, as will at once keep them distinct, and shew the law by which one part is derived from another.

This may even be verified in some of the most elementary problems of algebra, as in the two following formulas before alluded to, which include all the cases in the transformation

of binomial factors and the extraction of the roots of equations.

Formula I.

$$\begin{array}{l|l|l|l}
{}_{0}P= \quad B & {}_{0}Q= \quad C & {}_{0}R= \quad D & \&c. \\
{}_{1}P= {}_{0}P+Aa' & {}_{1}Q= {}_{0}Q+ \; {}_{1}Pa'' & {}_{1}R= {}_{0}R+ \; Qa''' & \&c. \\
{}_{2}P= {}_{1}P+Ab' & {}_{2}Q= {}_{1}Q+ \; {}_{2}Pb'' & \cdot \;\cdot \;\cdot \;\cdot \;\cdot \;\cdot \;\cdot & \\
{}_{3}P= {}_{2}P+Ac' & \cdot \;\cdot \;\cdot \;\cdot \;\cdot \;\cdot \;\cdot & {}_{n-2}R={}_{n-3}R+{}_{n-2}Ql''' & \&c. \\
\cdot \;\cdot \;\cdot \;\cdot \;\cdot \;\cdot & {}_{n-1}Q={}_{n-2}Q+{}_{n-1}Pl'' & & \\
{}_{n}P={}_{n-1}P+Al' & & &
\end{array}$$

The above formula shews the rule for the transformation of the product

$$(x+a)(x+b)(x+c), \text{ \&c. to } n \text{ factors}$$

into a series of the form

$$((x+\alpha)(x+\beta)(x+\gamma), \text{ \&c. to } n \text{ factors}) + P((x+\alpha)(x+\beta), \text{ \&c. to } n-1 \text{ factors}) + Q((x+\alpha)(x+\beta), \text{ \&c. to } n-2 \text{ factors}) + \text{\&c.};$$

by making $A=1$, and B, C, D, &c. each equal to zero; a', b', c', &c. respectively equal to $a-\alpha$, $b-\beta$, $c-\gamma$, &c.; a'', b'', c'', &c. respectively equal to $b-\alpha$, $c-\beta$, $d-\gamma$, &c.; a''', b''', c''', &c. respectively equal to $c-\alpha$, $d-\beta$, $e-\gamma$, &c.; then the values of ${}_{n}P$, ${}_{n-1}Q$, ${}_{n-2}R$, &c. are those of the coefficients P, Q, R, &c. This formula includes the binomial theorem combinations of all degrees, &c.

This formula also exhibits the rule for the transformation of the equation $Ax^n+Bx^{n-1}+Cx^{n-2}+\ldots+Lx=N$ into another, of which the root shall be less than the root of the equation proposed by a given quantity a;

By making a', b', c', &c., a'', b'', c'', &c., a''', b''', c''', &c. each equal to a. In this case, as in the former, ${}_{n}P$, ${}_{n-1}Q$, ${}_{n-2}R$, are the coefficients of the second, third, fourth, &c. terms of the transformed equation.

The formula also shows how the value of the series or compound quantity $Ax^n+Bx^{n-1}+Cx^{n-2}+\ldots+Lx$ may be found in the easiest manner in terms of the quantities a, b, c, &c. when $x=a+b+c+\ldots+l$, or in terms of x itself.

The following formula is expressed in the same manner as the preceding one, and is exceedingly useful in the transformation of quantities.

FORMULA II.

$$\begin{array}{l|l|l|l}
{}_1B = \quad Aa' & {}_1C = \quad {}_1Ba'' & {}_1D = \quad {}_1Ca''' & \&c. \\
{}_2B = {}_1B + Ab' & {}_2C = {}_1C + {}_2Bb'' & {}_2D = {}_1D + {}_2Cb''' & \&c. \\
{}_3B = {}_2B + Ac' & {}_3C = {}_2C + {}_3Bc'' & {}_3D = {}_2D + {}_3Cc''' & \&c. \\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \\
{}_mB = {}_{m-1}B + Al' & {}_mC = {}_{m-1}C + {}_mBl'' & {}_mD = {}_{m-1}D + {}_mCl''' & \&c.
\end{array}$$

One of the uses of this formula is to transform the fraction

$$\frac{1}{(x+\alpha)(x+\beta)(x+\gamma), \text{ \&c. to } m \text{ factors}}$$

into a series of other fractions of the form

$$\frac{1}{(x+a)(x+b), \text{ \&c. to } m \text{ factors}} + \frac{P}{(x+a)(x+b), \text{ \&c. to } m+1 \text{ factors}}$$
$$+ \frac{Q}{(x+a)(x+b), \text{ \&c. to } m+2 \text{ factors}} + \text{\&c.};$$

by making a', b', c', &c. respectively equal to $a-\alpha$, $b-\beta$, $c-\gamma$, &c.; a'', b'', c'', &c. respectively equal to $b-\alpha$, $c-\beta$, $d-\gamma$, &c.; a''', b''', c''', &c. respectively equal $c-\alpha$, $d-\beta$, $e-\gamma$, &c.; then the coefficients P, Q, R, &c. are the values of ${}_mB$, ${}_mC$, ${}_mD$, &c. found as directed by the formula.

Another use of this formula is to find all the orders of election of the quantities *aaa*, &c. *bbb*, &c. *ccc*, &c. and so on;

This is by making each of the quantities a', b', c', &c, respectively equal to a, b, c; each of the quantities a'', b'', c'', &c. respectively equal to b, c, d, &c.; each of the quantities a''', b''', c''', &c. respectively equal to c, d, e, &c.; and $A=1$. The first, second, third, &c. orders of election are then expressed by the values of ${}_mB$, ${}_mC$, ${}_mD$, &c.

Another use of this formula is to find the values of the terms of the product of $(1+a+a^2+a^3+\ldots a^n)$ $(1+b+b^2+b^3+\ldots+b^n)(1+c+c^2+c^3+\ldots+c^n)$, &c. to m factors,

By making the quantities in the formula as in the last application.

There are other applications of the two preceding formulas.

I shall now demonstrate the truth of Formula I, as it applies to the transformation of equations.

In the orders of figurate numbers, Let A, A, A, &c. represent the first $n+1$ consecutive terms of the first vertical column; ${}_0P$, ${}_1P$, ${}_2P$, &c. represent the first n consecutive terms of the second vertical column; ${}_0Q$, ${}_1Q$, ${}_2Q$, &c. the first $n-1$ consecutive terms of the third vertical column; and so on. Whence, by the construction of these numbers, we have

No. 1.

$$\begin{array}{ll|ll|ll|l}
{}_0P= & 1 & {}_0Q= & 1 & {}_0R= & 1 & \text{\&c.} \\
{}_1P= & {}_0P+A & {}_1Q= & {}_0Q+\ {}_1P & {}_1R= & {}_0R+\ {}_1Q & \\
{}_2P= & {}_1P+A & {}_2Q= & {}_1Q+\ {}_2P & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot & \\
{}_3P= & {}_2P+A & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot & {}_{n-2}R= & {}_{n-3}R+{}_{n-2}Q & \\
\cdot\ \cdot & \cdot\ \cdot\ \cdot & {}_{n-1}Q= & {}_{n-2}Q+{}_{n-1}P & & & \\
{}_nP= & {}_{n-1}P+A & & & & &
\end{array}$$

But every term in the second vertical column may be decomposed into the preceding order, with two terms in each order; every term of the third column into the preceding order, with three terms in each order; and so on.

Hence the value of ${}_nP$ will be the nth order of figurate numbers, with two terms; the value of ${}_{n-1}Q$ will be the $(n-1)$th order, with three terms in each; and so on.

No. 2.

$$\text{Whence } {}_nP=\frac{1^{n-1|1}}{1^{n-1|1}}+\frac{2^{n-1|1}}{1^{n-1|1}}$$

$${}_{n-1}Q=\frac{1^{n-2|1}}{1^{n-2|1}}+\frac{2^{n-2|1}}{1^{n-2|1}}+\frac{3^{n-2|1}}{1^{n-2|1}}$$

$${}_{n-2}R=\frac{1^{n-3|1}}{1^{n-3|1}}+\frac{2^{n-3|1}}{1^{n-3|1}}+\frac{3^{n-3|1}}{1^{n-3|1}}+\frac{4^{n-3|1}}{1^{n-3|1}}$$

&c. &c.

But the values of the coefficients B_2, C_2, D_2, page 25 of the Essay on Involution and Evolution, are as follow:

No. 3,

$$B_2=\frac{1^{n-1|1}}{1^{n-1|1}}B+\frac{2^{n-1|1}}{1^{n-1|1}}Aa$$

$$C_2=\frac{1^{n-2|1}}{1^{n-2|1}}C+\frac{2^{n-2|1}}{1^{n-2|1}}Ba+\frac{3^{n-2|1}}{1^{n-2|1}}Aa^2$$

$$D_2=\frac{1^{n-3|1}}{1^{n-3|1}}D+\frac{2^{n-3|1}}{1^{n-3|1}}Ca+\frac{3^{n-3|1}}{1^{n-3|1}}Ba^2+\frac{4^{n-3|1}}{1^{n-3|1}}Aa^3$$

&c, &c.

Now these values of B_2, C_2, D_2, &c. No. 3, are respectively the same as the values in No. 2, except that the terms have the coefficients A, B, C, &c. from left to right; but as this is the order of adding the preceding to the following values, and as any term in any former value has the exponent of a less by unity than that term in the following value which has the same letter, we shall have the following formula, instead of the figurate formula No. 1: viz.

No. 4.

$$\begin{array}{l|l|l|l}
{}_0P= B & {}_0Q= C & {}_0R= D & \text{\&c.} \\
{}_1P= {}_0P+Aa & {}_1Q= {}_0Q+{}_1Pa & {}_1R= {}_0R+{}_1Qa & \\
{}_2P= {}_1P+Aa & {}_2Q= {}_1Q+{}_2Pa & \cdots\cdots\cdots & \\
{}_3P= {}_2P+Aa & \cdots\cdots\cdots & {}_{n-2}R={}_{n-3}R+{}_{n-2}Qa & \\
\cdots\cdots & {}_{n-1}Q={}_{n-2}Q+{}_{n-1}Pa & & \\
{}_nP={}_{n-1}P+Aa & & &
\end{array}$$

Before I begin to show the rules, and the operations directed by them, it will be proper to exhibit the readiest methods of performing the minor parts of the process.

PROPOSITION I. Problem.

To multiply one number by a digit, and add the product, as each figure arises, to another number.

Case 1.—When both numbers have the same sign, *Multiply the first or right-hand figure of the multiplicand by the digit; add the first figure of the additive number to the product; write the figure in the unit's place of the sum for the first figure of the required number, and carry the tens as so many units. Again, multiply the second figure of the multiplicand by the digit; add the number carried, and the figure in the second place of the additive number, to the product; write the units of the sum in the second place of the required number. Proceed with each remaining number, one after the other, in the same manner.*

Case 2.—When the two numbers have different signs, *Multiply the first or right-hand figure of the multiplicand by the digit; subtract the first figure of the product from the first figure of the other number; write the figure belonging to the unit's place of the remainder for the first figure of the required number; add as many units to the figure, in the place of tens of the product, as the number of tens borrowed, which cannot exceed one ten, and carry the sum to the next product. Proceed with the remaining figures in the same manner.*

Examples.

1.—*Let it be required to multiply the number* 3885 *by the digit* 6, *and to add the product, as every figure is multiplied, to the number* 48321.

Say 6 times 5 is 30 (see the marginal operation), and 1 in the additive number is 31; set down 1 and carry 3. Again, say 6 times 8 is 48, and 3 carried is 51, and 2 in the second place of the additive number makes 53; set down 3 and carry 5. Again, say 6 times 3 is 18, and 5 carried is 23, and 8, the next digit of the additive number, is 31; set down 1 and carry 3; and so on.

$$\begin{array}{r|l} & 48321 \\ 3885 \times 6 & 71631 \end{array}$$

Then 71631 is the number required; that is, $71631 = 3885 \times 6 + 48321$.

2.—*Multiply the number* —791 *by* 3, *and add the product to* 195027.

$$\begin{array}{r|l} & 195027 \\ -791 \times 3 & 192654 \end{array}$$

Say 3 times 1 is 3 (see the marginal operation); subtract 3 from 7, and 4 remains. Again, say 3 times 9 is 27; and as nothing was carried, nor any increase, subtract 7 from 12, and there remains 5; then adding 2 in the place of tens of the product, 27, and 1 for the increase, makes 3 to carry. Again, say 3 times 7 is 21, and 3 to carry is 24; subtract 4 from 10, and there remains 6; then 2 in the place of tens, and 1 the increase, is 3; then, as there are no more products, it will be 3 from 5, and 2 remains; and bring down the remaining figures as they stand in the single number, and place them in the same order in the number required.

3.—*Multiply* 195027 *by* 3, *and add the product to* —14344773.

$$\begin{array}{r|l} & -14344773 \\ 195027 \times 3 & -13759692 \end{array}$$

Explanation as in the preceding Example.

PROPOSITION II. Problem.

To find the first figure of the root of an equation.

Find a near or estimated value by Problem iv, *or otherwise; arrange the coefficients of the given equation in a line, but detached from each other. Multiply the coefficient of the first term of the equation by the estimated figure of the root; add the product to the coefficient of the second term, and write the sum under the second term. Multiply the sum by the estimated figure of the root; add the product to the coefficient of the third term, and write the sum under the coefficient of the third term; and so on. Multiply the number under the coefficient of the last term by the estimated figure of the root; then if the product be less than the absolute number, the estimated figure is the first figure of the root; but if it is greater, repeat the process.*

EXAMPLES.

1.—*Find the first figure of the root of the quadratic equation* $3x^2+4x=1$.

Here the absolute number 1 is less than the coefficient 4 of the single power; therefore divide 1 by 4, and the first figure ·2 of the quotient is the estimate value of the first figure, which is thus proved:

3	4	1·
(·2)	4·6	·92

and since ·92 is less than 1, ·2 is the first figure of the root.

2.—*Find the first figure of the root of the cubic equation* $x^3+5x^2+7x=47$.

Here, as 47 is greater than 7, we must therefore take the nearest cube to 47, which is 27; let the root 3 be tried in the operation: thus,

1	5	7	47
(3)	8	31	93

3 is therefore too much. Let us try again with 2:

1	5	7	47
(2)	7	21	42
			5

therefore the first figure of the root is 2.

3.—*Find the first figure of the root of the biquadratic equation* $x^4+5x^3+4x^2+3x=105$.

Here, as the absolute number is much greater than 3, the coefficient of x, we must in this case take the nearest fourth power to 105; and this is 81, the root of which is 3: let us therefore try 3.

1	5	4	3	
(3)	8	28	87	261

3 is therefore too great: let us try 2; then

1	5	4	3	
(2)	7	18	39	78

2 is therefore the first figure of the root.

4.—*Find the first figure of the root of the biquadratic equation* $x^4 - 80x^3 + 1998x^2 - 14937x = -5000$.

Here, by dividing the absolute number —5000 by —14937, we get ·3, which is proved by the operation:

1	—80	1998	—14937	—5000
(·3)	—797	197409	—14344·773	—4303·4319

therefore 3 is the first figure of the root.

PROPOSITION III. Problem.

To transform an equation, so that the root of the transformed equation shall be less than the root of the given equation by the greatest digit possible, not exceeding the root.

Find the first figure of the root of the proposed equation as before, and let the process remain; call the numbers under the coefficient of the second, third, &c. terms the quadratic, cubic, &c. divisors.

Then, considering the row of divisors the first of a series of rows which have their terms so disposed that in every two consecutive rows the lower row may have one term less than the upper row, and every term of the lower row under the corresponding term of the upper row, and the number of the rows to be such that the last row may have one term.

Any term in any row will be found by multiplying the opposite term on the left by the root figure, and adding the product to the term, above the place of the term required for that term, which write down.

Proceed in this manner from the first to the last member of every row, observing that the first term of every row is the coefficient of the first term of the equation which is understood.

Then the last number in each respective column is the coefficient of the transformed equation, and the remainder is the absolute number of the new equation.

EXAMPLES.

1.—*Transform the equation* $3x^2+4x=1$ *into another, of which the root shall be less than that of the equation proposed by the greatest digit possible, not exceeding the root of the proposed equation.*

Here the first figure of the root will be found to be $\cdot 2$.

$$\begin{array}{llr} 3 & 4 & 1\cdot \\ (\cdot 2) & 4\cdot 6 & \cdot 92 \\ & 5\cdot 2 & \overline{\cdot 08} \end{array}$$

Therefore the new equation is $3x^2+5\cdot 2x=\cdot 08$.

2.—*Transform the cubic equation* $x^3+5x^2+7x=47$ *into another, of which the root shall be less than that of the one here proposed by the greatest digit possible, not exceeding the root.*

Here the first figure will be found to be **2.**

$$\begin{array}{lrrr} 1 & 5 & 7 & 47 \\ (2) & 7 & 21 & 42 \\ & 9 & 39 & \overline{5} \\ & 11 & & \end{array}$$

Therefore the transformed equation is $x^3+11x^2+39x=5$.

3.—*Transform the biquadratic equation*

$$x^4+5x^3+4x^2+3x=105$$

into another, of which the root shall be less than that of the one proposed by the greatest digit possible, not exceeding the root.

$$\begin{array}{lrrrr} 1 & 5 & 4 & 3 & 105 \\ (2) & 7 & 18 & 39 & 78 \\ & 9 & 36 & 111 & \overline{27} \\ & 11 & 58 & & \\ & 13 & & & \end{array}$$

Therefore the transformed equation is

$$x^4+13x^3+58x^2+111x=27.$$

4.—*Transform the equation*

$$x^4-80x^3+1998x^2-14937x=-5000$$

into another, of which the root shall be less than the root of the given equation by the greatest digit possible, not exceeding that root.

K*

1	—80	1998	—14937	—5000
(·3)	—79·7	1974·09	—14344·773	—4303·4319
	—79·4	1950·27	—13759·692	—696·5681
	—79·1	1926·54		
	—78·8			

Therefore the transformed equation is

$$x^4-788x^3+1926\cdot54x^2-13759\cdot692x=-696\cdot5681.$$

PROPOSITION IV. Problem.

To extract the root of the general equation
$$Ax^n+Bx^{n-1}+Cx^{n-2}+\ldots Lx=N.$$

Transform the given equation into another, of which its root shall be less than the root of the proposed equation by the greatest figure of the root of the given equation.

Proceed with this last equation as before, and so on.

Examples.

1.—*Extract the root of the quadratic equation* $3x^2+4x=1$.

3	4	1·	(·2
(·2)	4·6	·92	
	5·2	·08	

Whence the first transformed equation is $x^2+5\cdot2x=\cdot08$.

Again,	3	5·2	·08	(·01
	(·01)	5·23	·0523	
		5·26	·0277	

Whence the second transformed equation is

$$3x^2+5\cdot26x=\cdot0277.$$

Again,	3	5·26	·0277	(005
	(·005)	5·275	·026375	
		5·290	·001325	

Whence the third transformed equation is

$$3x^2+5\cdot290x=\cdot001325$$

and so on.

Therefore the value of x, as far as we have gone, is $x=\cdot215$.

By uniting these separate operations we shall have the continued operation

Here the continued operation is extended two steps further than the number of transformed equations.

The decimal point is discontinued after the first step, the law for placing the numbers being obvious.

Here the constant figures in the second line of each step are left out, retaining only two figures.

```
      3     4
(2)   46  | )1.  (·21525
      52  |  ·92
            ---------
(1)   523   |  800
       26   |  523
              ---------
(5)   5275   | 27700
        90   | 26375
               ---------
(2)   52906   |  132500
         12   |  105812
                ---------
(5)   529135   | 2668800
               | 2645675
                 -------
                 ··23125
```

2.—*Extract the root of the cubic equation* $x^3+5x^2+7x=47$.

Suppose in the third step of the work we wished to ascertain the figure of the root belonging to the next or fourth step. Annex a cipher to the remainder 159872 of the third step, and divide the number 1598720 by the coefficient 416832, and the quotient 3 is the next figure of the root. The same is to be understood in the discovery of every new figure.

```
 1      5    |7                       ............)47(2.123
(2)     7....|21 divisor ....:                    42
        9    |39                  : ...... )5000
       11    | :                  :         4011
      111... |4011 divisor......  :   ......)989000
(1)   112    |4123                :          829128
      113    |  :                 :      ...)159872000
      1132.  |414564 divisor......:       :   125151867
(2)   1134   |416832                      :    34720133
      1136   |   :                        :      &c.
(3)   11363  |41717289 divisor........... :
      &c.    |   &c.
```

The above operation, when curtailed by leaving out the repeating figures in all the lines except that which stands at the head of each class in the line of the divisor, and performing the divisions upon the same principle, will be as follows :—

```
   5      | 7
   7 .....|21                    47  (2·123
   9      |39                    5
  11      |                       989
 111....  |4011                   159872
   2      | 123                    34720133
   3      |
 1132.    |414564
    4     |  6832
    6     |
 11363    |41717289
```

3.—Extract the cube root of the number 3 to four places of figures in the root.

```
 1    0      |0                    3(1·442
(1)   1      |1                    1
      2      |3                    ----
      3      |                     2000
      34.... |436                  1744
(4)   38     |588                  ------
      42     |                      256000
      424... |60496                 241984
      428    |62208                  --------
      432    |                       14016000
(2)   4322   |6229444                12458888
                                     --------
                                      1557112
```

The same by the common method of extracting the cube root.

```
        3(1·442        10        10      4
         1             10        16      4
   300)2000            ---       ---    --
       1200            100       160    16
        480            3         3       4
         64            ---       ---    --
       ----            300       480    64
       1744            =====================
 58800) 256000          140       140   4
        235200          140       16    4
          6720          ---       ---   --
            64          56        84    16
        ------          14        14    4
        241984          -----     ----  --
6220800)14016000        19600     2240  64
        12441600          3         3
           17280        -----     ----
               8        58800     6720
        --------        =====================
        12458888          1440      1440  2
        --------          1440        4   2
         1557112          ----      ----  -
                          576       5760  4
                          576         3   2
                          144       -----  -
                        -------     17280 8
                        2073600
                            3
                        -------
                        6220800
```

The above work, by the general method, contains only 100 figures; and the last, by the common method of extracting the cube root, contains 225 figures.

4.—*Extract the root of* $x^4+5x^3+4x^2+3x=105$.

```
5          4            3     ....................)105(2·217
7......... 18.......... 39...:                     78
9          36           111       ................)270000
11         58            :       :                  246256
13          :            :       :               ..)237440000
132........6064.........123128...:              :    136453781
 4         332          35792       :           ...)1009862190000
 6         604            :         :              963079632921
 8                        :         :              46782557079
1381.......661781......136453781....:
 2          3163        7116944           :
 3          4546           :              :
 4                         :              :
13847......66551529...137582804703........:
 54         648507      8049344252
 61         745534
 68
```

Having proceeded thus far with the operation, and there being no hopes of its terminating, yet as it is desirable to extend the root with as little trouble as possible, this object will be accomplished by observing the following rule:

In the last transformed equation, cut off one figure from the right of the coefficient of the single power, or trial divisor; two from the coefficient of the second power; three from the coefficient of the third power; and so on.

Then in every two consecutive numbers in the same line begin with the figure on the left-hand of the line of the left-hand number, and observe what would be carried from the figure on the right, and add the product to the remaining part on the left of the right-hand number. Proceed from left to right until all the numbers are found.

The reason of the rule is obvious; for, since the first number of any succeeding class, in any step, has as many figures to the left of the coefficient above it as the number of terms to that place after the first not included, it follows, that if n figures be cut off from the right of the number under any coefficient, and as many figures be cut off from the left of that coefficient as the remaining number of figures on the left of the number under it, the number of figures remaining on the right of the coefficient will be equal to the exponent of the power to which that coefficient belongs.

```
(3)     13|868   667455|34   13804934425|2   ................46782557079(3
                    497|.............6936917|.......:       41420810751
                    539|              8939515|     ...............)5361746328(3
                    581|                                    4142741940
         |013
(3)      |         6675|85......1380913979|0......   ............)1219004387(8
                       |89         34007|                     1104751479
                       |8                       ..........)114252908(8
  |000|                                                        110475617
                     66|75......... 138093934|7.........  ...........)3777291(2
                       |7                 468|                  2761891
                       |7                       138094)1015400(7
                   |66............ 13809452|0..............   ....  966661
                                          7|                        48739(3
                             1380945|7 ................:            41428
                                                                     7311(5
                                                                     6904
                                                                      407(2
                                                                      276
                                                                      131(9
                                                                      124
                                                                      ..7(7
                                                                        7
```

Extract the root of $x^4-80x^3+1998x^2-14937x=-5000$, *which is the only complete example Mr. Holdred has given to his last improved method in the Supplement, and let the work be performed by Mr. Nicholson's rule before given.*

```
      —80·       1998          —14937      ................)—5000·0000(35
      —797   ...197409...... —14344773...:              —43034319
(3)   —794       195027        —13759692      ...............)—69656810000
      —791       192654                                    —68317809375
      —788
(5)   —7875...19226025... —13663561875...:  ..............)—1339000625(09
      —7870      19186675      —13567628500                 —1220931477
      —7865      19147350                   ..............)—118069148(8
      —7860                                                 —108512233
|—007|860   1914|66|13  —135659053|0|2...:  ..............)—9556915(7
            1914|            —135641822|                     —9494704
(09)        1914|                                  —135638)—62211(04
                                                            —54255
            19|14........—13564029|1..........:             )—7956)5
            19|              —13563876|                      —6782
            19|                                             )—1174(8
          |19............—1356386|3.............:            —1085
                          —1356385|                             89

                          —135638|5                 Whence x=·3509870458.
```

The work here by my rule contains fewer figures by 138 than that of Mr. Holdred's, in page 56 of his Supplement; but even this work may be considerably reduced, as follows:

OPERATION.

<table>
<tr><td>—80</td><td>1998</td><td>—14937</td><td>—5000 (3</td></tr>
<tr><td>—797</td><td>197409</td><td>— 4344773</td><td>—6965681 (5</td></tr>
<tr><td>— 94</td><td>5027</td><td>— 3735992</td><td>—1339000625(09</td></tr>
<tr><td>— 91</td><td>2654</td><td></td><td>—118069148(8</td></tr>
<tr><td>— 88</td><td></td><td></td><td>—9556915(7</td></tr>
<tr><td>—7875</td><td>19226025</td><td>—13663561875</td><td>—62211(04</td></tr>
<tr><td>— 70</td><td>86675</td><td>— 567628500</td><td>—7956(5</td></tr>
<tr><td>— 65</td><td>47350</td><td></td><td>—1174(8</td></tr>
<tr><td>— 60</td><td></td><td></td><td>—89</td></tr>
<tr><td>|00|—78|60</td><td>1914|66|43</td><td>—13565905|02</td><td></td></tr>
<tr><td></td><td></td><td>— 641822</td><td></td></tr>
<tr><td></td><td></td><td>—13564029|1</td><td>Whence $x = \cdot 3509870458$.</td></tr>
<tr><td></td><td></td><td>— 3876|</td><td></td></tr>
<tr><td></td><td></td><td>—1396386|3</td><td></td></tr>
<tr><td></td><td></td><td>— 85|</td><td></td></tr>
<tr><td></td><td></td><td>—135638|5</td><td></td></tr>
</table>

In general, in the first column, as every new step arises, one figure more may be left out of every curtailed line than in the preceding step, and making each left-hand curtailed number to consist of one figure more than the number at the head of the class on the right-hand.

In the above Example, the work of the subtractions is performed upon the same principle as that by which the other numbers are found; viz. by multiplying the divisor by the root figure, and subtracting the product at the same time.

This elegant method of multiplying and subtracting at the same time was first introduced into this country by Dr. Hutton and Mr. Bonnycastle, in their excellent treatises on arithmetic, in the operation of division.

I have now, I believe, shown the superiority of my demonstrations and methods to those of Mr. Holdred's. I am confident that he never had any clear notions of treating the subject. The sum of the whole is, that he submitted his work to me for my opinion, which I not only gave to him freely, but also communicated many important improvements, which he adopted as his own. The approbation which he requested me to give of his rule in writing,

and which he published in his Prospectus, has been of considerable use in bringing his work into public notice. I have all along given him the credit of being the first person who had any just conceptions of the method; but when I saw that his intention was to extract all from me that he could, in order that his work might appear to the greatest advantage, and that it was his determination to take no notice of what I had done, it must be allowed that he has treated me with ingratitude for my services and for the use which he has made of my name; and when I perceived that the grounds upon which I proceeded, and the demonstrations founded upon them, were entirely different from his, I only wished to adjoin my improvements to his Tract in my own name: had this request been granted, no separate work of mine should ever have appeared in competition with his. It was my wish to serve him, and not to do him an injury. It may be that Mr. Holdred's Tract may be purchased without mine; but no one can read my Essay without being desirous of procuring his also; and then, by a fair comparison of the two works, I have not the smallest doubt but that the reader will coincide with me in stating that what I have done are real improvements; and, if so, my claims will be fully justified.

J. Compton, Printer, Middle Street, Cloth Fair, London.

FACTORIALS.

DEFINITION.

IF in the combination of several quantities used as the factors of an algebraic product, the difference between every two adjacent factors be the same, such a product is called a factorial*.

NOTATION.

As the parts which determine a factorial consist of the first factor, the number of factors, and the common difference, and as factorials become powers when the common difference is zero, the symbol expressing a factorial ought to have such a relation to that of a power, that, when the difference is zero, the remaining parts of the symbol may indicate a power, according to the usual notation.

Therefore, if the first factor be written in the manner of the root, with a superior on the right, indicating the number of factors, and on the right of the superior, in the same line, another superior, making the common difference, be written, with a line between the two superiors, the symbol thus formed will indicate the factorial required; but if the factors decrease, place a short line over the right-hand superior.

---

* Thus, $1 \cdot 2 \cdot 3 \cdot 4$, or $7 \cdot 5 \cdot 3$, or $x(x+1)(x+2)(x+3)$, or $x(x-1)(x-2)(x-3)$ is a factorial; but $1 \cdot 3 \cdot 7 \cdot 9$ is not a factorial; since the difference between the first and second adjacent numbers is not equal to the difference between the second and third.

Thus, the factorial $x(x+1)(x+2)(x+3)$ may either be represented by $x^{4|1}$ or by $(x+3)^{4|\overline{1}}$; so that every factorial may be represented in two different ways, as $1\cdot2\cdot3\cdot4\cdot5\cdot6$ may either be $1^{6|1}$ or $6^{6|\overline{1}}$; and so on.

PROPOSITION I. PROBLEM.

To resolve a factorial into two factorial factors, so that one of them may have any given number of factors less than the given one.

Make the first factor of one of the factorial factors equal to the first factor of the given factorial [and its exponent equal to the difference of the two given exponents]; then make the first factor of the second factorial factor, whose exponent is only given, equal to the sum of the first factor, and the product made of the common difference and the exponent of the first factorial factor.

EXAMPLES.

Ex. I.—*Resolve* $(m+1)^{n|1}$ *into two factorial factors, so that one of them may have the given exponent* 1, *or to find the last factor.*

By rule

$$(m+1)^{n|1}=(m+1)^{n-1|1}\times(m+n)^{1|1}=(m+1)^{n-1|1}\times(m+n).$$

Ex. II.—*Find the last factor of the factorial* $1^{n|1}$.

This is the same as to resolve the factorial $1^{m|1}$ into two factorial factors, so that one of them may have the given exponent 1; whence, by rule, $1^{n|1}=1^{n-1|1}\times n$.

Ex. III.—*Resolve the factorial* $m^{n-1|1}$ *into two factorial factors, so that one of them may have the given exponent* m—1.

By rule $m^{n-1|1}=m^{n-m|1}\times n^{m-1|1}$.

Ex. IV.—*Resolve the factorial* $1^{n-1|1}$ *into factorial factors, so that the second may have the given exponent* n—m.

By rule $1^{n-1|1} = 1^{m-1|1} \times m^{n-m|1}$

PROPOSITION II. Problem.

A factorial being given, to reverse the order of the factors.

Find the last factor of the factorial by Prop. 2; then write the last factor instead of the first, and the exponent and common difference the same as in the given factorial, and mark the common difference with the negative sign —.

Example.—*Reverse the order of the factors of the factorial*

$$(m-3)^{4|1}.$$

Here the last factor is m; whence $(m-3)^{4|1} = m^{4|\bar{1}}$.

FIGURATE NUMBERS.

DEFINITIONS.

Def. I.—In any number of series

a',	b',	c',	d',	e',	f',
a'',	b'',	c'',	d'',	e'',	f'',
a''',	b''',	c''',	d''',	e''',	f''',
&c.		&c.		&c.	

placed in due order, if n be the number of any series, and x the number of the term in the nth series; and if the xth term of the nth series be expressed by $\frac{x^{n-1|1}}{\lfloor n-1|1}$, each series is called an order of figurate numbers.

Corollary 1.—Hence, by this definition, the nth order will be

$$\frac{1^{n-1|1}}{1^{n-1|1}},\ \frac{2^{n-1|1}}{1^{n-1|1}},\ \frac{3^{n-1|1}}{1^{n-1|1}},\ \frac{4^{n-1|1}}{1^{n-1|1}} \cdots \frac{m^{n-1|1}}{1^{n-1|1}}$$

m being here the number of the last term.

Corollary 2.—Hence the first term of every order of figurate numbers is unity.

Corollary 3.—If n be 1, the series will denote the first order of figurate numbers; but in this case $n-1=0$; therefore all the exponents will be zero; that is, as in powers, each term will have no factors; therefore the first order of figurate numbers will be a series of units 1, 1, 1, 1, 1, &c.

Corollary 4.—If n be 2, the series will denote the second order of figurate numbers; but in this case $n-1=1$; so that the numerator and denominator of each term will only have one factor; therefore the second order of figurate numbers are the natural series 1, 2, 3, 4, 5, &c.

Def. II.—The mth term of the first, second, third, &c. order is called the mth vertical column; that is,

$$\frac{m^{1-1|1}}{1^{1-1|1}},\ \frac{m^{2-1|1}}{1^{2-1|1}},\ \frac{m^{3-1|1}}{1^{3-1|1}},\ \frac{m^{4-1|1}}{1^{4-1|1}},\ \text{\&c. or } 1,\ m,\ \frac{m^{2|1}}{1^{2|1}},\ \frac{m^{3|1}}{1^{3|1}}, \text{\&c.}$$

is the mth vertical column.

Def. III.—The $(m+1)$th term of the first order, the mth term of the second order, the $(m-1)$th term of the third order, and so on, are called the mth diagonal series; that is,

$$\frac{(m+1)^{1-1|1}}{1^{1-1|1}},\ \frac{m^{2-1|1}}{1^{2-1|1}},\ \frac{(m-1)^{3-1|1}}{1^{3-1|1}},\ \frac{(m-2)^{4-1|1}}{1^{4-1|1}},\ \text{\&c.}$$

$$\text{or } 1,\ m,\ \frac{(m-1)^{2|1}}{1^{2|1}},\ \frac{(m-2)^{3|1}}{1^{3|1}},\ \text{\&c.}$$

PROPOSITION I. Theorem.

The $(m+1)$th term of the $(n+1)$th order is equal to the mth term of the $(n+1)$th order plus the $(m+1)$th term of the nth order.

For $\frac{(m+1)^{n|1}}{1^{n|1}}$ is the $(m+1)$th term of the $(n+1)$th order; but $(m+1)^{n|1}=(m+1)^{n-1|1}\times(m+n)$, [see Ex. 1, Prop. i, Factorials] and $(m+1)^{n-1|1}\times(m+n)=m^{n|1}+n(m+1)^{n-1|1}$, by multiplying the reduced factorial by each part of the last factor.

Again, $1^{n|1}=1^{n-1|1}\times n$ [see Ex. 2, Prop. i, Factorials]; whence $\frac{(m+1)^{n|1}}{1^{n|1}}=\frac{m^{n|1}+n(m+1)^{n-1|1}}{1^{n-1|1}\times n}=\frac{m^{n|1}}{1^{n|1}}+\frac{(m+1)^{n-1|1}}{1^{n-1|1}}$.

PROPOSITION II. Theorem.

In any two consecutive orders of figurate numbers, the sum of m terms of the antecedent is equal to the mth term of the consequent order.

For, let $1, b, c, d,$ &c.

$1, \beta, \gamma, \delta,$ &c.

be any two consecutive orders; then, by the last proposition,

$$1+b=\beta$$
$$\beta+c=\gamma$$
$$\gamma+d=\delta$$
&c. &c.

The sum of these equations is

$$1+\beta+\gamma+b+c+d=\beta+\gamma+\delta.$$

Take away the common quantities β, γ, and there will remain

$$1+b+c+d=\delta. \quad \text{Q.E.D.}$$

Corollary.—Hence, because the first order is a series of units, all the following orders may be derived from each other*.

PROPOSITION III. Theorem.

The *m*th term of the *n*th order of figurate numbers is equal to the *n*th term of the *m*th order.

For $\frac{m^{n-1|1}}{1^{n-1|1}}$ is the *m*th term of the *n*th order. Now the factorial $m^{n-1|1}$ may be resolved into two factorial factors, so that one of them may have the given exponent $m-1$; therefore $m^{n-1|1}=m^{n-m|1}\times n^{m-1|1}$. See Ex. 3, Prop. i, Factorials.

Likewise, the factorial $1^{n-1|1}$ may be resolved into two factorial factors, so that one of them may have the given exponent $n-m$; therefore $1^{n-1|1}=1^{m-1|1}\times m^{n-m|1}$ (see Ex. 4, Prop. i, Factorials);

whence $\frac{m^{n-1|1}}{1^{n-1|1}}=\frac{m^{n-m|1}\times n^{m-1|1}}{1^{m-1|1}\times m^{n-m|1}}=\frac{n^{m-1|1}}{1^{m-1|1}}$. Q.E.D.

Corollary.—Hence one expression of figurate numbers may easily be converted into another equivalent expression by the following

Rule:—Add unity to the exponent of the numerator of the given expression, and it will give the first factor of the numerator of the new expression; take unity from the first

---

* Thus,

1st order....	1,	1,	1,	1,	1,	1,	&c.
2d order....	1,	2,	3,	4,	5,	6,	&c.
3d order....	1,	3,	6,	10,	15,	21,	&c.
4th order...	1,	4,	10,	20,	35,	56,	&c.
5th order...	1,	5,	15,	35,	70,	126,	&c.
&c.	&c.		&c.		&c.		

factor of the numerator, and the remainder will be the exponent of the new numerator.

The exponent of the denominator of the new factorial is the same as its numerator, and the first factor the same as that of the given expression.

PROPOSITION IV. Theorem.

The terms of the nth order of figurate numbers are each respectively equal to each of the terms of the nth vertical column.

For, since (by Prop. iii) the mth term of the nth order is equal to the nth term of the mth order, we have

$$\frac{m^{n-1|1}}{1^{n-1|1}} = \frac{n^{m-1|1}}{1^{m-1|1}};$$

but the nth term of the mth order is the mth term of the nth vertical column; therefore, making m successively equal to 1, 2, 3, &c., then the terms of the nth order will be respectively equal to the terms of the nth vertical column*.

PROPOSITION V. Theorem.

The sum of x terms of the vth vertical column is equal to the $(v+1)$th term of the xth order of figurate numbers.

For (by Prop. iv) each of the x terms of the vth order, taken progressively, are respectively equal to each of the x terms of the vth vertical column taken progressively.

And since (by Prop. ii) the sum of x terms of the vth order is equal to the xth term of the $(v+1)$th order,

* Thus, 1, $\frac{2^{n-1|1}}{1^{n-1|1}}$, $\frac{3^{n-1|1}}{1^{n-1|1}}$, $\frac{4^{n-1|1}}{1^{n-1|1}}$, &c. respectively $=1$, n, $\frac{n^{2|1}}{1^{2|1}}$, $\frac{n^{3|1}}{1^{3|1}}$, &c.

Therefore the sum of x terms of the vth vertical column is equal to the xth term of the $(v+1)$th order.

But (by Prop. iii) the xth term of the $(v+1)$th order is equal to the $(v+1)$th term of the xth order;

Whence the sum of x terms of the vth vertical column is equal to the $(v+1)$th term of the xth order. Q.E.D.

Corollary 1.—Hence the second, third, fourth, &c. terms, to the $(m+1)$th term inclusive, of the nth order, are respectively equal to the sum of all the first terms, the sum of all the second terms, the sum of all the third terms, &c. to the sum of all the m terms, inclusive of the first consecutive n orders of figurate numbers.

Corollary 2.—Hence if the first term of any order of figurate numbers be taken away, the remaining terms may be decomposed into as many of the first consecutive orders as there are units in the exponent of the order to be decomposed, with as many terms in each of the orders thus decomposed as there are terms in the part which was to be decomposed*.

---

* Thus, in the nth order $\frac{1^{n-1|1}}{1^{n-1|1}}, \frac{2^{n-1|1}}{1^{n-1|1}}, \frac{3^{n-1|1}}{1^{n-1|1}} \ldots \frac{(m+1)^{n-1|1}}{1^{n-1|1}}$, by taking away the first term $\frac{1^{n-1|1}}{1^{n-1|1}}=1$, the remaining terms $\frac{2^{n-1|1}}{1^{n-1|1}}, \frac{3^{n-1|1}}{1^{n-1|1}} \ldots \frac{(m+1)^{n-1|1}}{1^{n-1|1}}$ may be decomposed into the following n consecutive orders:

1, 1, 1, 1, 1, to m terms
1, 2, 3, 4, 5, to m terms
1, 3, 6, 10, 15, to m terms
. . . .
$\frac{1^{n-1|1}}{1^{n-1|1}}, \frac{2^{n-1|1}}{1^{n-1|1}}, \frac{3^{n-1|1}}{1^{n-1|1}} \ldots \frac{m^{n-1|1}}{1^{n-1|1}}$

PROPOSITION IV. THEOREM.

The mth diagonal series of figurate numbers is of the form

$$1,\ m,\ \frac{m^{2|1}}{1^{2|1}},\ \frac{m^{3|1}}{1^{3|1}},\ \frac{m^{4|1}}{1^{4|1}},\ \&c.$$

For, by Definition 3, the mth diagonal series of figurate numbers is

$$1,\ m,\ \frac{(m-1)^{2|1}}{1^{2|1}},\ \frac{(m-2)^{3|1}}{1^{3|1}},\ \frac{(m-3)^{4|1}}{1^{4|1}},\ \&c.$$

Now, reversing the order of the factors in the numerators, we have

$$1,\ m,\ \frac{m^{2|1}}{1^{2|1}},\ \frac{m^{3|1}}{1^{3|1}},\ \frac{m^{4|1}}{1^{4|1}},\ \&c.\quad \text{Q.E.D.}$$

PROPOSITION V. THEOREM.

In any two consecutive diagonal series, if the nth term of the antecedent be added to the $(n+1)$th term, the sum will be the $(n+1)$th term of the consequent series.

This is evident from Prop. i; for, in any two consecutive orders of figurate numbers, if the mth term of the consequent order be added to the $(m+1)$th term of the antecedent order, the sum will be the $(m+1)$th term of the consequent order.

But the mth term of the consequent order, and the $(m+1)$th term of the antecedent orders, may be any two consecutive terms x and $x+1$ of a diagonal series; likewise, the $(m+1)$th term of the consequent order is the $(x+1)$th term of the next diagonal series.

Whence, let $1,\ B,\ C,\ D,\ E,$ &c.
and $1,\ B',\ C',\ D',\ E',$ &c.

be any two consecutive diagonal series; then will

$$1+B=B'$$
$$B+C=C'$$
$$C+D=D'$$
$$D+E=E' \quad \text{Q.E.D.}$$
&c. &c.

SCHOLIUM.—It is a most curious circumstance, that since the mth term of the $(n+1)$th order is equal to the $(n+1)$th term of the mth order, and that since the terms of the nth order, taken progressively, are each respectively equal to the terms of the nth vertical column taken progressively, therefore the terms of the mth order, taken progressively, are each respectively equal to each of the terms of the mth vertical column, taken progressively; whence it follows, that the sums of all the four series

$$\frac{1^{n-1|1}}{1^{n-1|1}}+\frac{2^{n-1|1}}{1^{n-1|1}}+\frac{3^{n-1|1}}{1^{n-1|1}}+\ldots+\frac{m^{n-1|1}}{1^{n-1|1}}$$

$$\frac{1^{m-2|1}}{1^{m-2|1}}+\frac{2^{m-2|1}}{1^{m-2|1}}+\frac{3^{m-2|1}}{1^{m-2|1}}+\ldots+\frac{(n+1)^{m-2|1}}{1^{m-2|1}}$$

$$\frac{n^{1-1|1}}{1^{1-1|1}}+\frac{n^{2-1|1}}{1^{2-1|1}}+\frac{n^{3-1|1}}{1^{3-1|1}}+\ldots+\frac{n^{m-1|1}}{1^{m-1|1}}$$

$$\text{and}\ \frac{(m-1)^{1-1|1}}{1^{1-1|1}}+\frac{(m-1)^{2-1|1}}{1^{2-1|1}}+\frac{(m-1)^{3-1|1}}{1^{3-1|1}}+\ldots+\frac{(m-1)^{n|1}}{1^{n|1}}$$

are equal to each other, and that the sum of each series is equal to $\frac{m^{n|1}}{1^{n|1}}$, or equal to $\frac{(n+1)^{m-1|1}}{1^{m-1|1}}$.

N.B.—Each two series that are coupled together have every two corresponding terms equal, the one series being an order of figurate numbers, and the other the vertical column of the same number.

ARITHMETICAL EQUIVALENTS.

Arithmetical operations not unfrequently occur, which require the aggregation of quantities affected with unlike signs. Such is often the case in extracting the roots of equations, in various applications of logarithms, in ascertaining the arithmetical value of an infinite series, &c. &c.; and much time is necessarily consumed, from the obligation we are under of adding apart the quantities affected with like signs, and then subtracting the less result from the greater, in order to find the sum of the whole.

This inconvenience has always been felt in logarithms; and to escape it we generally make use of the arithmetical complement of the logarithm affected with the negative sign, and then add the whole nearly in the usual manner.

The *arithmetical complement* of a negative number is the difference between that number and another which has unity for the first figure on the left-hand, followed by as many ciphers as there are digits in the proposed negative number.

This artifice may, with no small advantage, be introduced into some of the ordinary processes of arithmetic; but we must extend considerably the theory usually given, and employ, not merely the arithmetical complement of any given negative number, but an expression actually equivalent to the number itself, and which, for that reason, may be called the *arithmetical equivalent*.

To explain this generally, let $-a$ represent any number affected with a negative sign, and let $-b$ be another negative number, consisting of unity, followed by as many ciphers as there are places of figures in $-a$; let it be required to find such an affirmative part x, that $-a$ shall be equal to the aggregate of $-b$ and x. This proposition, expressed algebraically, is $-a=-b+x$; whence $x=b-a$: this value

being called c, we shall have $-a=-b+c$; then $-b+c$ is the *arithmetical equivalent.*

It is evident that the affirmative part c will have the same number of places of figures which the negative number $-a$ has; therefore, by prefixing a negative unit to the left-hand of the number expressed by c, we shall form the true numeral expression of the arithmetical equivalent.

For example, let -31416 be the proposed negative number; it is required to find an expression equivalent to it, such that all the figures except the first on the left shall be affirmative; then, according to the principle and rule now stated, $100000-31416=68584$, the affirmative part; therefore the mixed number, equivalent to -31416, is $\bar{1}68584$, or $-31416=\bar{1}68584$. This expression has the properties desired.

The subtraction may be performed at sight, and is attended with scarcely any more trouble than that of writing the figures down.

For, instead of taking the difference between -31416 and 100000, we might subtract every figure of -31416, except that on the right-hand, from 9, and the right-hand figure 6 from 10; then the remainders being set down in their proper order, prefix a negative unit. It will, therefore, be indifferent in whatever direction we proceed; suppose, then, that we proceed from right to left; thus, 6 from 10 four remains, 1 from 9 eight remains, 4 from 9 five remains, 1 from 9 eight remains, and 3 from 9 six remains; therefore the arithmetical equivalent is $\bar{1}68584$, the same as before.

From the same consideration, any two numbers, of which one is affirmative and the other negative, being given, such that the negative one may have as many ciphers on the right-hand as the affirmative one contains places of figures, may be reduced to their arithmetical equivalent.

In this case we have only to write the affirmative part as it is given; then to subtract the significant figures of the negative number, as before; then prefixing the remainders in their proper order on the left-hand of the affirmative number, and on the left-hand of all the negative unit, and the expression so formed will be the arithmetical equivalent.

Thus, let ·65327 be the decimal part of the logarithm of a fractional number, and the index be -7; then 7 from 10 three remains; therefore the arithmetical equivalent is $\bar{1}3{\cdot}65327$; or, if -700000 and 65327 be the two whole numbers, as stated, their arithmetical equivalent will be $\bar{1}365327$.

Any arithmetical operation may be performed upon these equivalents; and if we keep steadily in mind that they are compound expressions, partly affirmative and partly negative, and apply the rules respecting the signs as given in the Elements of Algebra, we shall experience little difficulty in the use of them.

But as I have all along endeavoured to explain whatever I have introduced into this work in the clearest possible manner, and as I have employed the arithmetical equivalents in several of the examples on the extraction of roots, I shall illustrate the theory of them by a few examples.

PROPOSITION I. PROBLEM.

To find an arithmetical equivalent for a given negative number.

Rule.—Subtract each figure of the given number from 9, except the figure in the unit's place, which subtract from 10, and on the left-hand of the result write 1, with the negative sign over it.

EXAMPLES.

$$-9864=\bar{1}0136,\quad -762=\bar{1}238,\quad -3543=\bar{1}6457.$$

PROPOSITION II. PROBLEM.

To assign the negative number represented by any arithmetical equivalent.

Rule.—Subtract the figure in the unit's place from 10, and all the rest from 9, except the one surmounted by the negative sign, which efface altogether.

EXAMPLES.

$\bar{1}7623 = -2377$, $\bar{1}3861 = -6139$, $\bar{1}425 = -575$.

Observation.—It frequently happens that we have equivalents in which the digit on the left-hand is greater than unity; then, instead of proceeding as in Prop. ii, after subtracting the figure in the unit's place from 10, and the rest from 9, we must diminish the first negative figure, in proceeding to the left, by 1, as $\bar{4}3861 = -36139$.

PROPOSITION III.

To find the aggregate of numbers affected with unlike signs.

Rule.—Find the arithmetical equivalents of the negative numbers, and add them and the affirmative numbers together, deducting the negative figures in each column as they occur.

EXAMPLE.

Add 7854, 31416, —734, 65321, and —2965 *together.*

Common Method.		By Arithmetical Equivalents.
7854	734	7854
31416	2965	31416
65321	3699	$\bar{1}266$
104591		65321
3699		$\bar{1}7035$
100892		100892

In casting up the fourth column from the right, I say, 7 and 1 carried make 8, and 5 make 13, and minus 1 makes 12 and 1 makes 13, &c. &c.

PROPOSITION IV.

To subtract arithmetical equivalents.

Rule.—Proceed exactly as in algebraic subtraction; that is, conceive that the signs of the numbers to be subtracted are changed, and then add those that have like signs, and subtract those that are unlike.

EXAMPLES.

$\bar{1}364$	$\bar{1}876$	153	$\bar{3}141$
$\bar{1}876$	$\bar{1}364$	$\bar{2}67$	5236
$\bar{1}488$	$+512$	$+286$	$\bar{9}905$

The same Examples according to the usual method:

-636	-124	153	-2859
-124	-636	-133	$+5236$
-512	$+512$	$+286$	-8095

PROPOSITION V.

To multiply an arithmetical equivalent by a given number.

Rule.—Multiply in the usual manner, observing only that the number which we carry from the last affirmative figure must be *deducted* from the negative product.

EXAMPLES.

$\bar{1}5632$	$\bar{5}89367$	$\bar{2}734$
9	7	18
$\bar{4}0688$	$\bar{2}\bar{9}25569$	$\bar{1}\bar{1}872$
		$\bar{2}734$
		$\bar{3}7212$

An arithmetical equivalent containing several figures affected with the negative sign, may be transformed into another with only a single negative digit, by subtracting the first negative digit on the right-hand from 10, and all the subsequent ones except the last from 9, as in Prop. i, adding a unit to the last negative digit.

Thus, $\bar{2}\bar{9}25569 = \bar{3}125569$; and, upon the same principle, -6321 is successively equal to $\bar{6}\bar{3}\bar{3}9$, or $\bar{6}\bar{4}79$, or $\bar{7}679$, or $\bar{1}3679$.

PROPOSITION VI.

To divide an arithmetical equivalent by a given number.

Rule.—If the negative part is exactly divisible by the divisor, write the quotient below, with the negative sign over it, and proceed with the affirmative part as in common division. But if the negative part is not exactly divisible, increase it till it becomes so, and set down the quotient with the negative sign; and whatever number we add to make the negative part divisible, we must add an equal number of tens to the left-hand figure of the affirmative part, and then proceed as in ordinary division.

Examples.

$4)\bar{8}7832$ $5)\bar{7}8540$ $\frac{\bar{8}6432}{19}=\bar{1}6128.$
$\bar{2}1958$ $\bar{2}7708$

For, by the rule, we have, first,

$$\frac{\bar{8}+\bar{1}\bar{1}}{19}=\frac{\bar{1}\bar{9}}{19}=\bar{1};$$

and then

$$11\times10=110$$

```
       6432
19)116432(6128
   114
    24
    19
     53
     38
     152
     152
     ...
```

Whence the quotient is $\bar{1}6128$. And the reason of this rule is sufficiently obvious. For $\bar{8}6432=-80000+6432$ $=-80000-110000+110000+6432=-190000+116432$; and, dividing by 19, we have $\frac{-190000}{19}+\frac{116432}{19}=-10000$ $+6128=\bar{1}6128$, as before.

J. Compton, Printer, Middle Street, Cloth Fair, London.